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Studies on Magneto-Hydrodynamic Waves and other Anisotropic Wave Motions

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STUDIES ON MAGNETO-HYDRODYNAMIC WAVES AND OTHER ANISOTROPIC WAVE MOTIONS

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CONTENTS

There are two separate but closely interwoven strands of argument in this paper; one mainly mathematical, and one mainly physical.

The mathematical strand begins with a method of asymptotically evaluating Fourier integrals in many dimensions, for large values of their arguments. This is used to investigate partial differential equations in four variables, x, y, z and t, which are linear with constant coefficients, but which may be of any order and represent wave motions that are anisotropic or dispersive or both. It gives the asymptotic behaviour (at large distances) of solutions of these equations, representing waves generated by a source of finite or infinitesimal spatial extent. The paper concentrates particularly on sources of fixed frequency, and solutions satisfying the radiation condition; but an Appendix is devoted to waves generated by a source of finite duration in an initially quiescent medium, and to unstable systems. The mathematical results are given a partial physical interpretation by arguments determining the velocity of energy propagation in a plane wave traversing an anisotropic medium. These show, among other facts not generally realized, that even for non-dispersive (e.g. elastic) waves, the energy propagation velocity is not in general normal to the wave fronts, although its component normal to them is the phase velocity.

The second, mainly physical, strand of argument starts from the important and striking property of magneto-hydrodynamic waves in an incompressible, inviscid and perfectly conducting medium, of propagation in one direction only—a given disturbance propagates only along the magnetic lines of force which pass through it, and therefore suffers no attenuation with distance. There are cases of astrophysical importance where densities are so low that attenuation due to collisional effects—for example, electrical resistivity—should be negligible over relevant length scales. We therefore ask how far the effects of a non-collisional nature which are neglected in the simple theory, particularly compressibility and Hall current, would alter the unidirectional, attenuation-less propagation of the

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waves. These effects have been included previously in magneto-hydrodynamic wave theory, but the directional distribution of waves from a local source was not obtained. This problem explains the need for the mathematical theory just described, and gives a comprehensive illustration of its application.

1. Introduction

The elementary theory of magneto-hydrodynamic waves in an incompressible, perfectly conducting fluid (Alfvén 1942, 1950; Walén 1944; Spitzer 1956; Cowling 1957) indicates as a distinctive property of these waves that they do not spread out three-dimensionally around a source of disturbances, giving 'spherical attenuation' (intensity diminishing as the inverse square of distance from the source); instead, their propagation is purely one-dimensional, along the magnetic lines of force, and hence without attenuation.

For example, if the undisturbed magnetic field is $(B_0, 0, 0)$, in the x-direction, then the velocity vector $\mathbf{v}$ satisfies

$$\frac{\partial^2 \mathbf{v}}{\partial t^2} = a_1^2 \frac{\partial^2 \mathbf{v}}{\partial x^2}, \tag{1}$$

representing the propagation of disturbances purely in the x-direction, at the Alfvén velocity $a_1 = B_0/\surd(4\pi\rho)$, where ρ is the density and the units are e.m.u.

One may think of the transverse motions as propagated like waves on a stretched string, since a magnetic tube of force of unit area has mass ρ per unit length and tension $B^2/4\pi$ (coupled with a hydrostatic pressure $B^2/8\pi$, whose variations are however balanced by those of the gas pressure). The longitudinal motions satisfy the same equation (1) because they are directly coupled to transverse motions by the equation of continuity, div $\mathbf{v} = 0$.

The inclusion of a finite electrical conductivity produces exponential attenuation of the waves, but in many cosmical applications the conductivity may be presumed great enough for this attenuation to be negligibly slow. Therefore, their lack of purely geometrical attenuation makes magneto-hydrodynamic waves a convenient source of astrophysical explanations: a disturbance at one point may originate from a disturbance at some far distant point, which has travelled between the two along a line of magnetic force.

The object of this paper is to study how this distinctive property of magneto-hydrodynamic waves is modified by the effects of:

(*a*) compressibility;

(*b*) the use of a more realistic equation for current in a plasma.

Magneto-hydrodynamic waves have already been much studied in the presence of both these effects, especially (*a*) (Aström 1951; Herlofson 1950; van de Hulst 1951; Lundquist 1952, etc.). This paper, however, directs attention to a particular issue: what is the directional distribution of the waves produced by a local disturbance of given frequency? This gives a quantitative measure of such departure as there may be from pure one-dimensional propagation.

In these problems it is convenient to specify the disturbance by means of three quantities: the dilatation $\Delta = \text{div}\,\mathbf{v}$, the x-component of vorticity

$$\xi = \frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \tag{2}$$

(where the suffixes denote components), and the x-component of rate-of-strain $\Gamma = \partial v_x/\partial x$. These quantities completely determine $\mathbf{v}$ in infinite space, because first Γ determines v_x; and, secondly, $\Delta - \Gamma$ and ξ determine (v_y, v_z), whose two-dimensional divergence and curl they are.

When compressibility alone is taken into account, it is found that ξ continues to satisfy the one-dimensional equation

$$\frac{\partial^2 \xi}{\partial t^2} = a_1^2 \frac{\partial^2 \xi}{\partial x^2}, \tag{3}$$

although Γ and Δ satisfy simultaneous equations of more complicated, three-dimensional type, leading to spherical attenuation.

Therefore, any local disturbance does in part get propagated one-dimensionally, the part in question being the component of vorticity along the magnetic lines of force. Far away, where Γ and Δ have been attenuated to values small compared with ξ, the disturbance becomes a purely two-dimensional ($\Gamma = 0$), solenoidal ($\Delta = 0$), vortex motion. Instead of being convected with the fluid as such a disturbance would be in an ordinary conservative field, it propagates (in both directions) along the magnetic lines of force.

Next, when the ratio of the Alfvén velocity a_1 to the sound speed a_0 becomes small, one would expect the incompressible-flow approximation to become more and more accurate, at least as regards the propagation of the longitudinal disturbance Γ. In fact, we find that although Γ does suffer spherical attenuation, in the sense that its amplitude decreases with distance r like r^{-1}, a more appropriate phrase would be 'conical attenuation', in that its values are small in this case except within a narrow cone with apex at the disturbance and axis along the magnetic line of force.

When the further complication (*b*) is taken into account, the equations are modified, in a manner whose physical interpretation is that the magnetic lines of force are frozen into the electron gas, rather than into the gas as a whole. In this case the whole disturbance is found to suffer conical attenuation, spreading out within a cone whose angle is $\sin^{-1}(\omega/\omega_i)$, where ω is the frequency of the disturbance and ω_i is the gyro-frequency of the ions in the magnetic field. Accordingly, it is only for frequencies ω very low compared with ω_i that the falling-off of intensity of a disturbance as it propagates along magnetic lines of force will be small.

To make the deductions from the equations which are required in these problems, it is necessary to have a convenient technique for obtaining the asymptotic form at large distances of the fundamental (point-source) solution of a complicated system of coupled anisotropic wave equations. The method for this purpose which is developed in §§ 3 to 6 is believed to be novel, and may be found useful also in problems of anisotropic wave propagation in elasticity, optics, etc. [*Note added in proof.* Buchwald (1959) has now applied it to elasticity theory.] The fundamental solution is written down as a threefold Fourier integral, whose asymptotic behaviour is then evaluated in terms of the locus of singularities of the integrand, by use of ideas described, for example, by Lighthill (1958). The result can be expressed in a simple geometrical form which helps one to visualize its implications.

To add to the possibilities of use of this method in other fields of application, the main results on asymptotic behaviour of threefold Fourier integrals, and of the solutions of systems of equations of anisotropic, and possibly dispersive, wave propagation, are expressed in the form of two theorems, which hold under definite sets of conditions, and from whose proofs one can get the necessary ideas to tackle problems under different sets of conditions—some of these ideas being discussed in § 5. The geometrical interpretation of the theorems is given in detail in a separate section (§ 4); this material is not all original, but is essential to an understanding of the subject.

The results become clearer physically from a study of the velocity of energy propagation in a plane wave $u = a \exp[i(\omega t + \alpha x + \beta y + \gamma z)]$, which, in any system that admits such a wave for each (α, β, γ), the frequency ω varying with (α, β, γ), is determined in appendix A as $-(\partial\omega/\partial\alpha, \partial\omega/\partial\beta, \partial\omega/\partial\gamma)$. This leads to an interpretation of theorem 2 as saying that energy propagates purely radially from any source—even though the resulting wave crests may have complicated, cuspidal shapes (see figures 4 and 7) and lie exclusively within some cone.

The paper deals mainly with waves generated by sources of fixed frequency, but the effect of sources of finite duration is investigated in appendix B. The results are applied to magneto-hydrodynamics waves, which the modification (*b*) mentioned above renders dispersive, at the end of § 9.

The material of §§ 3 to 6 and the two appendixes was not published separately, because the author preferred to expound this mathematical work in the context of the physical problem, which gives such a clear reason for asking the questions and such a clear illustration of the method of getting the answers.

2. Magneto-hydrodynamic waves in a compressible, perfectly conducting fluid

We begin by writing down the equations of motion of a compressible fluid in a magnetic field $\mathbf{B}$, in electromagnetic units and with all terms of a dissipative character (due to finite conductivity, to viscosity and to heat conduction) neglected. The hydrodynamical equations are

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{v}) = 0, \tag{4}$$

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}.\nabla\mathbf{v}\right) = -\nabla p + \frac{(\operatorname{curl}\mathbf{B})\wedge\mathbf{B}}{4\pi}, \tag{5}$$

$$\frac{\partial p}{\partial t} + \mathbf{v}.\nabla p = a^2\left(\frac{\partial \rho}{\partial t} + \mathbf{v}.\nabla\rho\right), \tag{6}$$

where ρ is the density, $\mathbf{v}$ the velocity vector, p the pressure, and a^2 the derivative of p with respect to ρ at constant entropy (so that a is the sound speed). The equation for the rate of change of $\mathbf{B}$ is taken as

$$\frac{\partial \mathbf{B}}{\partial t} = \operatorname{curl}(\mathbf{v}\wedge\mathbf{B}), \tag{7}$$

which is the equation for the classical 'perfectly conducting fluid', whose basis will be considered critically in § 8.

When the departures of ρ, $\mathbf{v}$, p, a and $\mathbf{B}$ from uniform values ρ_0, 0, p_0, a_0 and $\mathbf{B}_0$ (where $\mathbf{B}_0$ is a constant vector) are regarded as so small that their squares and products are negligible, the equations (4) to (7) simplify to

$$\left.\begin{aligned} \frac{\partial \rho}{\partial t} &= -\rho_0 \operatorname{div}\mathbf{v}, \quad \frac{\partial \mathbf{v}}{\partial t} = -\frac{1}{\rho_0}\nabla p + \frac{(\operatorname{curl}\mathbf{B})\wedge\mathbf{B}_0}{4\pi\rho_0}, \\ \frac{\partial p}{\partial t} &= a_0^2\frac{\partial \rho}{\partial t}, \quad \frac{\partial \mathbf{B}}{\partial t} = \operatorname{curl}(\mathbf{v}\wedge\mathbf{B}_0). \end{aligned}\right\} \tag{8}$$

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Differentiating the second equation with respect to t, and using the rest to eliminate all the variables but $\mathbf{v}$, we obtain

$$\frac{\partial^2 \mathbf{v}}{\partial t^2} = a_0^2 \operatorname{grad} \operatorname{div} \mathbf{v} + \frac{\{\operatorname{curl} \operatorname{curl} (\mathbf{v} \wedge \mathbf{B}_0)\} \wedge \mathbf{B}_0}{4\pi\rho_0}. \tag{9}$$

When $\mathbf{B}_0 = 0$ this implies the familiar result that div $\mathbf{v}$ propagates at the speed of sound, a_0, while curl $\mathbf{v}$ remains unchanged with time.

In the presence of a non-zero undisturbed field $\mathbf{B}_0$, it is convenient to choose one of the co-ordinate axes, say, the x-axis, parallel to $\mathbf{B}_0$. Then (9) becomes

$$\frac{\partial^2 \mathbf{v}}{\partial t^2} = a_0^2 \operatorname{grad} \operatorname{div} \mathbf{v} + a_1^2 \left(0,\ \nabla^2 v_y + \frac{\partial \xi}{\partial z},\ \nabla^2 v_z - \frac{\partial \xi}{\partial y}\right), \tag{10}$$

where the components of $\mathbf{v}$ have been written as (v_x, v_y, v_z), and $\xi = \partial v_z/\partial y - \partial v_y/\partial z$ is the x-component of the vorticity curl $\mathbf{v}$. In (10), a_1 is the Alfvén velocity $B_0(4\pi\rho)^{-\frac{1}{2}}$.

The system (10) of three coupled equations for v_x, v_y and v_z, which is of the sixth order, can be reduced in complexity by a new choice of dependent variables. Thus, by writing down the x-component of the curl of equation (10), we obtain for ξ itself the simple equation

$$\frac{\partial^2 \xi}{\partial t^2} = a_1^2 \left(\nabla^2 \xi - \frac{\partial^2 \xi}{\partial y^2} - \frac{\partial^2 \xi}{\partial z^2}\right) = a_1^2 \frac{\partial^2 \xi}{\partial x^2}, \tag{11}$$

whose physical implications were noted in § 1. Other variables satisfying simplified equations are

$$\Delta = \operatorname{div} \mathbf{v}, \quad \Gamma = \frac{\partial v_x}{\partial x}. \tag{12}$$

From (10), we have

$$\frac{\partial^2 \Delta}{\partial t^2} = a_0^2 \nabla^2 \Delta + a_1^2 \nabla^2 (\Delta - \Gamma), \tag{13}$$

and

$$\frac{\partial^2 \Gamma}{\partial t^2} = a_0^2 \frac{\partial^2 \Delta}{\partial x^2}, \tag{14}$$

two coupled equations forming a reasonably tractable system of the fourth order.

The three quantities ξ, Δ and Γ determine $\mathbf{v}$ completely in infinite space, since Γ determines v_x while the equations

$$\xi = \frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}, \quad \Delta - \Gamma = \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \tag{15}$$

determine v_y and v_z. It may be noted that although ξ is propagated one-dimensionally, along the magnetic lines of force, no other component of the motion is; for example, the y- and z-components of vorticity, η and ζ, satisfy

$$\frac{\partial^2 \eta}{\partial t^2} = a_1^2 \left(\frac{\partial^2 \eta}{\partial x^2} - \frac{\partial^2 \Delta}{\partial x \partial z}\right), \quad \frac{\partial^2 \zeta}{\partial t^2} = a_1^2 \left(\frac{\partial^2 \zeta}{\partial x^2} + \frac{\partial^2 \Delta}{\partial x \partial y}\right), \tag{16}$$

showing that only for incompressible flow ($\Delta = 0$) do they satisfy the same equation as ξ; in a compressible fluid their oscillations are coupled to those of Δ.

We now study in detail the angular distribution of waves from a localized disturbance, concentrating on the distributions of Γ and Δ, since the solution of equation (11) for ξ is trivial. From (13) and (14), Γ satisfies

$$\frac{\partial^2}{\partial t^2} \left\{\frac{\partial^2 \Gamma}{\partial t^2} - (a_0^2 + a_1^2)\, \nabla^2 \Gamma\right\} + a_0^2 a_1^2 \frac{\partial^2}{\partial x^2} \nabla^2 \Gamma = 0, \tag{17}$$

and the same equation is satisfied by Δ.

Now, to represent the radiation from a localized disturbance, equation (17) must be solved with a right-hand side which vanishes outside a finite region. This might represent the action of hydrodynamic sources, or of external ponderomotive or electromotive forces; or, simply, of the non-linear terms in the original equations, operating in some central region where motions are too large for them to be neglected.

This right-hand side, or 'forcing function', is supposed to have a certain characteristic length-scale l and frequency ω, the relationship of whose product to a_0 and a_1 may be expected from acoustic theory to affect the radiation produced. A simple choice, for example, would be

$$A\,\mathrm{e}^{\mathrm{i}\omega t}\frac{\exp\left[-(x^2+y^2+z^2)/l^2\right]}{(l\sqrt{\pi})^3}. \tag{18}$$

However, there would also be interest in the limit of (18) as the length-scale l tends to 0 (see, for example, Lighthill 1958), namely

$$A\,\mathrm{e}^{\mathrm{i}\omega t}\,\delta(x)\,\delta(y)\,\delta(z), \tag{19}$$

and in combinations of derivatives of (18) or (19). For example, if a hydrodynamic point source, producing $q\,\mathrm{e}^{\mathrm{i}\omega t}$ units of mass per unit time, were present at the origin, then the first of equations (8) would have $q\,\mathrm{e}^{\mathrm{i}\omega t}\,\delta(x)\,\delta(y)\,\delta(z)$ on the right, and it follows that the right-hand side of (17) would become

$$\frac{a_0^2 q}{\rho_0}\,\mathrm{e}^{\mathrm{i}\omega t}\,\frac{\partial^2}{\partial x^2}\,(a_1^2\nabla^2+\omega^2)\,\delta(x)\,\delta(y)\,\delta(z). \tag{20}$$

Similarly, a distribution of such sources over a region of length-scale l would give a right-hand side of

$$\frac{a_0^2 q}{\rho_0}\,\mathrm{e}^{\mathrm{i}\omega t}\,\frac{\partial^2}{\partial x^2}\,(a_1^2\nabla^2+\omega^2)\,\frac{\exp\left[-(x^2+y^2+z^2)/l^2\right]}{(l\sqrt{\pi})^3} \tag{21}$$

to the equation for Γ, and one of

$$\frac{a_0^2 q}{\rho_0}\,\mathrm{e}^{\mathrm{i}\omega t}\,\nabla^2\left(a_1^2\frac{\partial^2}{\partial x^2}+\omega^2\right)\frac{\exp\left[-(x^2+y^2+z^2)/l^2\right]}{(l\sqrt{\pi})^3} \tag{22}$$

to that for Δ. The effect on equation (17) of applied forces can be similarly worked out. The method for obtaining the asymptotic solution, to be given in § 3, applies equally easily to all such simple forcing functions.

3. Asymptotic solution of linear partial differential equations with constant coefficients

The problem posed in § 2 is that of finding the asymptotic behaviour as $r\to\infty$ of the solution of an equation of the form

$$P\left(\frac{\partial^2}{\partial t^2},\frac{\partial^2}{\partial x^2},\frac{\partial^2}{\partial y^2},\frac{\partial^2}{\partial z^2}\right)u=\mathrm{e}^{\mathrm{i}\omega t}f(x,y,z), \tag{23}$$

where P is a polynomial and $f(x,y,z)$ is a function vanishing outside a restricted region. Such problems can be solved formally by writing f as a threefold Fourier integral,

$$f(x,y,z)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left[\mathrm{i}(\alpha x+\beta y+\gamma z)\right]F(\alpha,\beta,\gamma)\,\mathrm{d}\alpha\,\mathrm{d}\beta\,\mathrm{d}\gamma. \tag{24}$$

Then we have

$$u = e^{i\omega t} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left[i(\alpha x + \beta y + \gamma z)\right] U(\alpha, \beta, \gamma)\, d\alpha\, d\beta\, d\gamma, \tag{25}$$

where substitution in (23) gives $GU = F$, of which a solution is

$$U = F/G, \tag{26}$$

if we write

$$G = P(-\omega^2, -\alpha^2, -\beta^2, -\gamma^2). \tag{27}$$

To be sure, this is only a particular integral. The general solution is obtained by adding a complementary function, namely, a solution of $GU = 0$, which, according to the theory of generalized functions, can be written $H\delta(G)$, H as well as G being a function of α, β and γ. However, this multiplicity of the solutions disappears in physical problems because of the 'radiation condition', which states that all waves originate at the source and none 'come in from infinity'. It is convenient to put off discussing how to apply this condition mathematically until § 6. Here and in § 4 we investigate the solution (26); but the reader must remember that slight modifications to the results will have to be made, as a result of the arguments in § 6, whenever the solution which satisfies the radiation condition is required.

Accordingly, we now discuss in detail the problem (which also has other applications) of estimating the integral in (25) as $\surd(x^2+y^2+z^2) = r \to \infty$, when U takes the form (26) and G is a known polynomial (27) in α, β and γ. In vector notation, with $\mathbf{r} = (x, y, z)$ and $\mathbf{k} = (\alpha, \beta, \gamma)$, the required integral, representing the amplitude u_0 of the fluctuations of u, is

$$u_0 = \iiint \frac{F(\mathbf{k})}{G(\mathbf{k})}\, e^{i\mathbf{k}\cdot\mathbf{r}}\, d\mathbf{k}. \tag{28}$$

Now, the asymptotic behaviour of any Fourier integral can be expressed in terms of the singularities of the integrand (Lighthill 1958). On the other hand, the function F has no singularities, since its Fourier transform f vanishes asymptotically to high order. We may illustrate this point by noting that the values of F, corresponding to expressions (18), (19), (20), (21) and (22), are

$$\left.\begin{array}{l} \dfrac{A}{8\pi^3}\exp\left(-\tfrac{1}{4}k^2l^2\right), \quad \dfrac{A}{8\pi^3}, \quad \dfrac{a_0^2 q}{8\pi^3\rho_0}\alpha^2\left(a_1^2k^2-\omega^2\right), \\ \dfrac{a_0^2 q}{8\pi^3\rho_0}\alpha^2\left(a_1^2k^2-\omega^2\right)\exp\left(-\tfrac{1}{4}k^2l^2\right), \quad \dfrac{a_0^2 q}{8\pi^3\rho_0}k^2\left(a_1^2\alpha^2-\omega^2\right)\exp\left(-\tfrac{1}{4}k^2l^2\right), \end{array}\right\} \tag{29}$$

respectively.

Accordingly, the singularities of the integrand in (29) lie entirely on the surface

$$G(\alpha, \beta, \gamma) = 0, \tag{30}$$

which we shall call S; it is the 'wave-number surface', or locus of points (α, β, γ) such that a plane-wave solution $u = \exp\left[i(\omega t + \alpha x + \beta y + \gamma z)\right]$ of the equation with zero right-hand side ($Pu = 0$) exists. We can expect u_0 to be asymptotic to the integral over S of $e^{i\mathbf{k}\cdot\mathbf{r}}$ times some function of $\mathbf{k}$; and hence, by the 'principle of stationary phase', to a sum of contributions from just those points on S where the exponent $\mathbf{k}\cdot\mathbf{r}$ is stationary. These are the points where the normal to the surface S is parallel to $\mathbf{r}$.

The detailed result, which bears out this expectation, is best found by the following device. We consider the asymptotic form of (28) as the point $\mathbf{r}$ tends to infinity along a particular radius vector l through the origin. To do this most easily, we choose new co-ordinate axes such that l is the positive x-axis. Then, having derived the answer, we express it in a form invariant under rotation of axes, after which it must be correct even for the original co-ordinate system, in which l was a line arbitrarily chosen. We now carry out this procedure.

In the new axes, a point $\mathbf{r}$ on l has co-ordinates $(x, 0, 0)$, where $x > 0$, and (28) becomes

$$u = \int_{-\infty}^{\infty} \mathrm{d}\beta \int_{-\infty}^{\infty} \mathrm{d}\gamma \int_{-\infty}^{\infty} \frac{F(\alpha,\beta,\gamma)}{G(\alpha,\beta,\gamma)}\, \mathrm{e}^{\mathrm{i}\alpha x}\mathrm{d}\alpha. \tag{31}$$

The inner integral can be evaluated asymptotically, for fixed β and γ, by the method of treating single Fourier integrals given in Lighthill (1958, chapter 4). There is a contribution from each value of α where $G = 0$, so that (α, β, γ) is on the surface S.

If G has a simple zero for each of these α, the inner integral in (31) is equal to

$$\pi\mathrm{i} \sum_S \frac{F(\alpha,\beta,\gamma)}{G_\alpha(\alpha,\beta,\gamma)}\, \mathrm{e}^{\mathrm{i}\alpha x} + O(x^{-N}) \tag{32}$$

as $x \to \infty$, for any N, if the suffix α denotes partial differentiation and $\sum_S$ is a sum over those values of α such that (α, β, γ) is on S. (Readers more familiar with a complex-variable approach to Fourier integrals will recognize (32) as half of $(2\pi\mathrm{i})$ times the sum of the residues of the integrand at its poles on the real axis; which gives the asymptotic behaviour as $x \to +\infty$ of the Cauchy principal value of the integral.) It follows from (32) that to a high order of approximation

$$u_0 \sim \pi\mathrm{i} \iint_S \frac{F(\alpha,\beta,\gamma)}{G_\alpha(\alpha,\beta,\gamma)}\, \mathrm{e}^{\mathrm{i}\alpha x}\mathrm{d}\beta\,\mathrm{d}\gamma \tag{33}$$

as $x \to \infty$, the integral being over the whole surface S.

However, if we are prepared to accept a lower order of approximation, we can obtain a far simpler asymptotic form. This is achieved by the 'principle of stationary phase', which says that the integral (33) is asymptotic as $x \to \infty$ to a sum of contributions from points on S where α, the coefficient of $\mathrm{i}x$ in the exponential, is stationary. These are points

$$\mathbf{k}_1, \mathbf{k}_2, \ldots, \mathbf{k}_n, \tag{34}$$

where the normal to S is parallel to the x-axis.

Let the contribution from any one of these points, say $\mathbf{k}_m$, to the asymptotic form of (33) for large x, be u_m. To evaluate u_m, we make now a further specialization of the choice of axes, choosing the y- and z-axes in the principal directions of curvature at this point $\mathbf{k}_m$ (where the x-axis is already given to be normal to the surface). If the associated curvatures, κ_β and κ_γ, assumed non-zero for the time being, are taken positive where concave to the x-direction and negative where convex, then an approximate equation of the surface S near $\mathbf{k} = \mathbf{k}_m$ is

$$\alpha = \alpha_m + \tfrac{1}{2}\kappa\ (\beta-\beta_m)^2 + \tfrac{1}{2}\kappa_\gamma(\gamma-\gamma_m)^2. \tag{35}$$

The principle of stationary phase then tells us that the contribution u_m to the asymptotic form of (33) for large x will be

$$u_m = \pi\mathrm{i}\frac{F(\mathbf{k}_m)}{G_\alpha(\mathbf{k}_m)}\, \mathrm{e}^{\mathrm{i}\alpha_m x} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left[\tfrac{1}{2}\mathrm{i}\kappa_\beta(\beta-\beta_m)^2 x + \tfrac{1}{2}\mathrm{i}\kappa_\gamma(\gamma-\gamma_m)^2 x\right] \mathrm{d}\beta\,\mathrm{d}\gamma. \tag{36}$$

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But, for any real h except zero,

$$\int_{-\infty}^{\infty} \exp\,[\mathrm{i}h(\beta-\beta_m)^2]\,\mathrm{d}\beta = \exp\,(\tfrac{1}{4}\pi\mathrm{i}\,\mathrm{sgn}\,h)\,\sqrt{\left(\frac{\pi}{|h|}\right)}, \tag{37}$$

because if h is positive one may put $\beta-\beta_m$ equal to $e^{\frac{1}{4}\pi\mathrm{i}}\,t$ to evaluate the integral as a Gaussian error integral, whereas if h is negative the necessary substitution is $\beta-\beta_m = \mathrm{e}^{-\frac{1}{4}\pi\mathrm{i}}\,t$. Hence, (36) becomes

$$u_m = \pi\mathrm{i}\,\frac{F(\mathbf{k}_m)}{G_\alpha(\mathbf{k}_m)}\,\mathrm{e}^{\mathrm{i}\alpha_m x}\exp[\tfrac{1}{4}\pi\mathrm{i}(\mathrm{sgn}\,\kappa_\beta+\mathrm{sgn}\,\kappa_\gamma)]\,\frac{2\pi}{x\sqrt{|\kappa_\beta\,\kappa_\gamma|}}. \tag{38}$$

The error in (38), according to the procedure for expanding to higher terms about the position of stationary phase (Jeffreys & Jeffreys 1950) is $O(x^{-2})$. The argument given above is incomplete, as possible contributions from those values of β and γ, for which the equation $G = 0$ for α has double or multiple roots, have not been considered. These are points of S where $G_\alpha = 0$, and one might at first expect from (33) that u_0 would asymptotically contain contributions from these singularities, although to be sure the area element $\mathrm{d}\beta\,\mathrm{d}\gamma$ (which is area projected on to $\alpha = 0$) also becomes vanishingly small at such points. Actually, a careful investigation by the same methods shows that there is no asymptotic contribution from such points.

An alternative derivation of (38), which perhaps makes it more obvious that the only contributions to the asymptotic behaviour of u_0 are from the points $\mathbf{k}_m$ where the normal to S is in the x-direction, is by integrating (31) first with respect to β and γ, and then asymptotically estimating the single Fourier integral that remains by the method of Lighthill (1958, chapter 4). The function

$$\iint (F/G)\,\mathrm{d}\beta\,\mathrm{d}\gamma \tag{39}$$

can be shown to have a singularity, as a function of α, only for values such that the plane $\alpha = \text{const.}$ is tangent to the surface $G = 0$. This singularity, which is a logarithmic infinity or a simple discontinuity according as κ_β and κ_γ have the same or opposite signs, leads to the term in x^{-1} given in (38).

We now express (38) in a form invariant under rotation of axes. To do this we replace x by r (which equals x in the existing axes, since $\mathbf{r} = (x, 0, 0)$ and $x > 0$), G_α by $\pm|\nabla G|$ according as ∇G is in the direction of $\pm\mathbf{r}$ (in one of which directions it must be, since the normal to the surface $G = 0$ at $\mathbf{k} = \mathbf{k}_m$ is parallel to $\mathbf{r}$); $\alpha_m x$ by $\mathbf{k}_m.\mathbf{r}$ and $\kappa_\beta\kappa_\gamma$ by the Gaussian curvature K of the surface. The Gaussian curvature is defined as the product of the principal curvatures, being positive where they have the same sign (synclastic curvature) and negative where they have opposite signs (anticlastic curvature). It possesses a general expression (see (43), below) in terms of the derivatives of G.

The substitutions just mentioned give

$$u_m = \frac{2\pi^2}{r}\left\{\frac{CF\mathrm{e}^{\mathrm{i}\mathbf{k}.\mathbf{r}}}{|\nabla G|\sqrt{|K|}}\right\}_{\mathbf{k}=\mathbf{k}_m}, \tag{40}$$

where C at $\mathbf{k} = \mathbf{k}_m$ is (i) $\pm\mathrm{i}$ if $K < 0$ and ∇G is in the direction of $\pm\mathbf{r}$, and (ii) ± 1 if $K > 0$ and the surface is convex to the direction of $\pm\nabla G$. These values of C follow from the fact that $\mathrm{sgn}\,\kappa_\beta+\mathrm{sgn}\,\kappa_\gamma$ is zero if $K < 0$, while if $K > 0$ it is ± 2 according as the surface is concave to the direction $\pm\mathbf{r}$.

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Expression (40) is clearly unchanged if we go back to the original axes. Hence we have

THEOREM 1.
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{F(\alpha,\beta,\gamma)}{G(\alpha,\beta,\gamma)}\exp\left[\mathrm{i}(\alpha x+\beta y+\gamma z)\right]\mathrm{d}\alpha\,\mathrm{d}\beta\,\mathrm{d}\gamma$$
$$=\frac{2\pi^2}{r}\Sigma\frac{CF\exp\left[\mathrm{i}\,(\alpha x+\beta y+\gamma z)\right]}{|\nabla G|\sqrt{|K|}}+O\left(\frac{1}{r^2}\right) \tag{41}$$

as $r\to\infty$ along any radius vector l, if the sum Σ is over that set (assumed finite) of points (α,β,γ) of the surface $G=0$ where the normal to the surface is parallel to l; provided that the surface has non-zero Gaussian curvature K at each of these points, and C is (i) $\pm\mathrm{i}$ where $K<0$ and ∇G is in the direction of $\pm\mathbf{r}$, (ii) ± 1 where $K>0$ and the surface is convex to the direction of $\pm\nabla G$.

In particular, one solution of

$$P\left(\frac{\partial^2}{\partial t^2},\frac{\partial^2}{\partial x^2},\frac{\partial^2}{\partial y^2},\frac{\partial^2}{\partial z^2}\right)u=\mathrm{e}^{\mathrm{i}\omega t}f(x,y,z) \tag{23 bis}$$

is asymptotically given by (41) *if*

$$F(\alpha,\beta,\gamma)=\frac{1}{8\pi^3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(x,y,z)\exp\left[-\mathrm{i}(\alpha x+\beta y+\gamma z)\right]\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \tag{42}$$

(the inverse of equation (24)), *and*

$$G(\alpha,\beta,\gamma)=P(-\omega^2,-\alpha^2,-\beta^2,-\gamma^2). \tag{27 bis}$$

To evaluate (41) one may use the general expression for Gaussian curvature of a surface given by an equation $G=0$, namely,

$$K=\frac{\Sigma G_\alpha^2(G_{\beta\beta}G_{\gamma\gamma}-G_{\beta\gamma}^2)+2\Sigma G_\beta G_\gamma(G_{\alpha\beta}G_{\alpha\gamma}-G_{\alpha\alpha}G_{\beta\gamma})}{(\Sigma G_\alpha^2)^2}, \tag{43}$$

where the sums are with respect to cyclic permutation of α, β and γ. The numerator is the sum of nine terms, each formed by multiplying an element, for example $G_\beta G_\gamma$, of the matrix of products of first derivatives of G, with the corresponding co-factor of the matrix of second derivatives (typical element $G_{\beta\gamma}$). When the x-axis is an axis of symmetry, as in the magneto-hydrodynamic problems considered in this paper, G takes the form $f(\alpha^2,\beta^2+\gamma^2)$, and then with $b=\alpha^2$, $c=\beta^2+\gamma^2$ expression (43) reduces to

$$K=\frac{f_c\{2bc(f_{bb}f_c^2-2f_{bc}f_bf_c+f_{cc}f_b^2)+f_bf_c(bf_b+cf_c)\}}{(bf_b^2+cf_c^2)^2}, \tag{44}$$

while $|\nabla G|$ is $2\sqrt{(bf_b^2+cf_c^2)}$. (We note in parentheses that threefold Fourier integrals with axisymmetry are commonly treated by conversion into Hankel transforms. However, the author's experience is that such conversion hinders their asymptotic evaluation, which is more easily achieved, as here, on the original threefold integral, than on the twofold Hankel integral with its more complicated integrand.)

To determine analytically (as required in case (ii)) whether the surface $G=0$ is convex to the direction $\pm\nabla G$, we need find only whether

$$\Sigma G_{\alpha\alpha}(G_\beta^2+G_\gamma^2)-2\Sigma G_{\beta\gamma}G_\beta G_\gamma \gtrless 0. \tag{45}$$

The left-hand side of (45) is in fact $|\nabla G|^3$ times the 'mean curvature' (*sum* of the principal curvatures), measured positive if convex to the direction G increasing. In the axisymmetrical case, the equivalent condition is simply $f_c \gtrless 0$, because a surface of revolution is necessarily concave to the axis of symmetry at any point of synclastic curvature.

To conclude this section we note that the same method of asymptotically estimating Fourier integrals in many dimensions is available for singularities of the integrand other than the simple poles here discussed. The initial step from (31) to (33) is different, and follows the procedure described in Lighthill (1958, chapter 4). The subsequent work, using the geometry of the locus of singularities, is as before.

4. Geometrical interpretation of theorem 1 for general anisotropic wave motions

The expression (41) represents a wave motion (indicated by the exponential), or more precisely a combination of a finite number of different wave motions, subject to three-dimensional attenuation (indicated by the $1/r$ factor). Note that every point on the wave-number surface $G = 0$ produces a particular sinusoidal wave train in a particular direction. This direction is that of the normal to the surface at the point; but the setting and spacing of the waves is indicated by the wave-number vector $(\alpha, \beta, \gamma) = \mathbf{k}$: the wave crests are set at right angles to this vector, and the wavelength is 2π divided by its magnitude. In general, the directions of the wave-number vector $\mathbf{k}$ and of the normal to S are not the same, so that the waves found in any direction are not set at right angles to it.

Note that normals to S at more than one point may lie in some particular direction. In such a direction, waves of different settings and spacings can be superimposed on one another. To be sure, these remarks do not apply if the points are simply $\mathbf{k}$ and $(-\mathbf{k})$, which correspond to waves of the same setting and spacing; but, actually, it will be shown in § 6 that the solution which satisfies the radiation condition takes a form in which one only out of each pair $\pm\mathbf{k}$ of points on the wave-number surface appears.

It is of interest to determine the shape of wave crests and troughs, and indeed of all surfaces of constant phase,

$$\mathbf{k}\cdot\mathbf{r} = N, \tag{46}$$

where N is any constant. Now, in (46), $\mathbf{k}$ and $\mathbf{r}$ are related, since $\mathbf{r}$ must be parallel to the normal to S at the point $\mathbf{k}$. This means that $\mathbf{r}$ is a simple multiple of ∇G, specified by (46) as

$$\mathbf{r} = \frac{N}{\mathbf{k}\cdot\nabla G}\nabla G. \tag{47}$$

Figure 1 shows that equation (47) places $\mathbf{r}$ in the direction of the perpendicular OD from the origin to the tangent plane at $\mathbf{k}$, while making its magnitude

$$r = N/OD. \tag{48}$$

In geometrical language, $\mathbf{r}$ is the 'pole', with respect to a sphere Σ of radius $\sqrt{N}$ and centre the origin, of the tangent plane to S at the point $\mathbf{k}$. The locus $\bar{S}$ of such points, namely, poles of tangent planes, is known as the 'reciprocal polar' of the surface S with respect to Σ. The adjective 'reciprocal' reminds us that, conversely, the reciprocal polar of $\bar{S}$ is S; that is, poles of tangent planes to $\bar{S}$ lie on S (so that $\bar{S}$ can also be regarded as the envelope of the polar planes of points on S). To see this result, note that a tangent plane to $\bar{S}$ may be thought of as the 'join' of three neighbouring points; hence its pole is the 'meet' of their polars, which lies on S because they are three neighbouring planes tangent to it.

Actually, this converse result has a simple physical interpretation. For the wavelength, that is, the perpendicular distance between surfaces (46) for phases N which differ by 2π, is clearly equal to $(2\pi/N)$ times the length of the perpendicular from the origin to a tangent plane of $\bar{S}$. But this length is N/k, where k is the distance from the origin to the corresponding point of S (pole of the tangent plane). Thus the reciprocal-polar relationship between the wave-number surface and the surfaces of constant phase ensures that the wavelength is $2\pi/k$, and is necessary to ensure it if the latter surfaces are to be geometrically similar to one another.

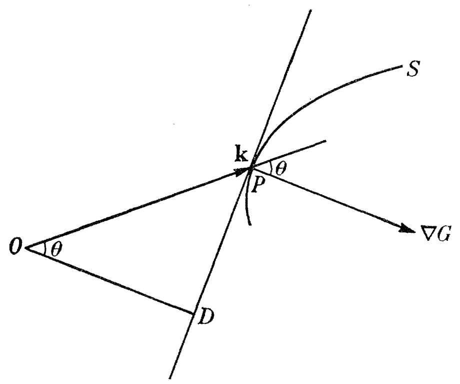

FIGURE 1. The vector ∇G, being normal to the surface $G = 0$, is parallel to **OD**, which makes an angle θ with **OP** $=$ **k**. Hence

$$\mathbf{k}.\nabla G = k\,|\nabla G|\cos\theta, \quad \text{and} \quad OD = OP\cos\theta = k\cos\theta = \frac{\mathbf{k}.\nabla G}{|\nabla G|}.$$

The amplitude variation in equation (41) can also be given a geometrical interpretation. Consider an elementary area $\mathrm{d}S$ of the wave-number surface. The radius vectors l, which are parallel to normals at points of this elementary area, fill a thin conical region whose cross-sectional area $\sim |K|\,r^2\,\mathrm{d}S$. Therefore, the dependence on $|K|^{-\frac{1}{2}}r^{-1}$ exhibited in (41) is consistent with the idea that the waves whose wave-number vectors **k** lie in $\mathrm{d}S$, and which are found in this conical region, satisfy a sort of 'conservation of intensity', the flux of $|u_0|^2$ carried by them being independent of the surface geometry, and equal in fact to

$$|u_0|^2\,|K|\,r^2\,\mathrm{d}S = \frac{4\pi^4 F^2}{|\nabla G|^2}\mathrm{d}S. \tag{49}$$

This idea is certainly a very vague one, but may help to render intelligible the appearance of K in theorem 1.

5. Cases of vanishing Gaussian curvature

Theorem 1 referred only to the effect of portions of the wave-number surface with non-zero Gaussian curvature K. We now make inquiries outside these limits.

First, consider a cylindrical portion C of the surface S. This can contribute to the asymptotic behaviour of u_0 along a radius vector l only if it is perpendicular to the generators of C. The special axes which can conveniently be used to determine the contribution have not only the x-axis along l but also the z-axis parallel to the generators. The normal to the surface is then in the x-direction not at isolated points but along a whole generator. If the principal curvatures are κ_β and 0 along this generator, the surface C has near it the approximate equation

$$\alpha = \alpha_m + \tfrac{1}{2}\kappa_\beta (\beta - \beta_m)^2, \tag{50}$$

with γ taking any value in a certain interval (a, b). The contribution u_m to the asymptotic form of (33) for large x is then

$$u_m = \pi \mathrm{i} \exp\left(\mathrm{i}\alpha_m x + \tfrac{1}{4}\pi \mathrm{i}\, \mathrm{sgn}\, \kappa_\beta\right) \left(\frac{2\pi}{x|\kappa_\beta|}\right)^{\frac{1}{2}} I_0, \tag{51}$$

where

$$I_0 = \int_a^b \frac{F(\alpha_m, \beta_m, \gamma)}{G_\alpha(\alpha_m, \beta_m, \gamma)}\, \mathrm{d}\gamma. \tag{52}$$

Equation (51) could easily be put into invariant form, since κ_β is equal to the mean curvature, for which an invariant expression was quoted at the end of § 3. However, the main thing to notice is simply the dependence on $r^{-\frac{1}{2}}$ instead of r^{-1}. This is characteristic of the fact that the waves discussed are propagated cylindrically, that is, only in directions at right angles to the generators of the cylinder.

It may be asked: how can there be an asymptotic contribution (51) along every such direction, but not along other directions however close? At first sight, there seems to be a discontinuity, a delta-function dependence on angle.

However, in the spatial variation of u_0 there is no discontinuity, as is seen if we consider a point slightly off the radius vector l, one which in the special axes introduced above has coordinates $(x, 0, z)$, so that it does not lie on a radius vector perpendicular to the generators. To obtain the value of u_0 at this point, a factor $\mathrm{e}^{\mathrm{i}\gamma z}$ must be included in the integrand of (33), so that the asymptotic form is (51) but with I_0 replaced by

$$I_z = \int_a^b \frac{F(\alpha_m, \beta_m, \gamma)}{G_\alpha(\alpha_m, \beta_m, \gamma)}\, \mathrm{e}^{\mathrm{i}\gamma z}\, \mathrm{d}\gamma. \tag{53}$$

Now, $I_z \to 0$ as $|z| \to \infty$ by the Riemann–Lebesgue theorem, so that there is only a fixed finite range of z within which u_m is significant.

This means that u_m represents a collimated cylindrical wave of finite width, lying everywhere within an infinite disk—a situation physically impossible for isotropic, but not for anisotropic, wave motions. It explains why the asymptotic contribution u_m vanishes on every radius vector not perpendicular to the axis of the disk, since any such radius vector stretches ultimately outside it.

In the extreme case when the portion C of surface is plane, an asymptotic contribution u_m arises from C only along the particular radius vector l which is at right angles to C. In axes such that the plane is $\alpha = \alpha_m$, this contribution is

$$u_m = \pi \mathrm{i}\, \mathrm{e}^{\mathrm{i}\alpha_m x} \iint_C \frac{F}{G_\alpha}\, \mathrm{d}\beta\, \mathrm{d}\gamma, \tag{54}$$

representing an unattenuated, plane wave. An investigation like that given in the cylindrical case shows that this is a collimated beam of finite diameter, which is why no asymptotic contribution appears on radii other than l.

Similar considerations to those described above for cylindrical portions of S apply also in the general case of any portion D on which the Gaussian curvature K is identically zero. Such a portion D of surface is 'developable', that is, deformable into a plane (without

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stretching). All conical as well as cylindrical surfaces are in this category, and the general developable surface is the envelope of a singly infinite family of planes P, each touching the surface along a whole straight line L.

Such a portion D of S makes a contribution to the asymptotic behaviour of u_0 along a radius vector l only if l is perpendicular to one of these tangent planes P. The contribution is obtained by taking the x-axis along l, as usual, and the z-axis in the direction of L, the line (or segment of a line) in which P touches D. The analysis is as above, except that κ_β now varies with γ along the line L (while κ_γ is still zero), so that instead of (51) we have

$$u_m = \pi i \left(\frac{2\pi}{x}\right)^{\frac{1}{2}} e^{i\alpha_m x} \int_L \frac{\exp\left[\frac{1}{4}\pi i \operatorname{sgn} \kappa_\beta(\gamma)\right]}{|\kappa_\beta(\gamma)|^{\frac{1}{2}}} \frac{F}{G_\alpha} d\gamma. \tag{55}$$

As before, we have attenuation like $r^{-\frac{1}{2}}$ along a singly infinite set of radii (those perpendicular to the tangent planes of D), and within a certain bounded distance of the conical surface formed by these radii.

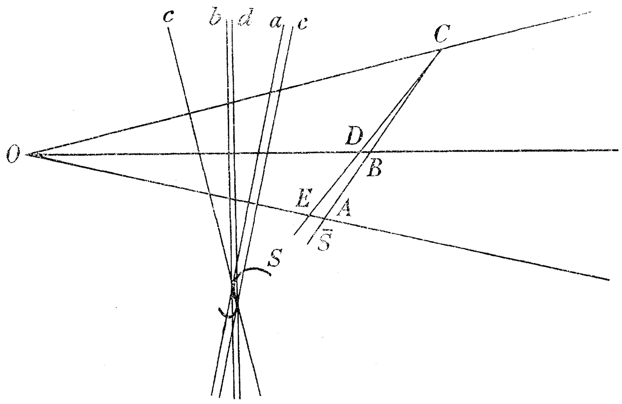

FIGURE 2. Successive tangents, a, b, c, d, e to the plane curve S (where c is the tangent at its point of inflexion) have poles A, B, C, D, E. The locus, S, of these poles has a cusp at C.

We pass next to cases when the Gaussian curvature vanishes, not throughout some portion of S, but only on a curve M, which divides the wave-number surface into synclastic ($K > 0$) and anticlastic ($K < 0$) regions. Such a curve on which K has a first-order zero is sometimes called a parabolic curve; we prefer to call it monoclastic (locus of points where the surface is singly curved). In this case theorem 1 specifies the asymptotic form of u_0 on all radii except those which are in the direction of the normal to S at some point of the monoclastic curve M.

The shape of a surface of constant phase thus predicted, namely, the reciprocal polar of S, has an 'edge of regression' ('cuspidal edge') corresponding to the monoclastic curve. Figure 2 illustrates the two-dimensional form of this result, showing how the reciprocal polar of a plane curve has a cusp corresponding to any point of inflexion (point of zero curvature). The general result is most easily seen if the equation of the surface be written as $\gamma = g(\alpha, \beta)$, when the Gaussian curvature (43) becomes

$$K = \frac{g_{\alpha\alpha} g_{\beta\beta} - g_{\alpha\beta}^2}{(1 + g_\alpha^2 + g_\beta^2)^2}, \tag{56}$$

and the parametric equation of the reciprocal polar (47) becomes

$$\mathbf{r} = N \frac{(g_\alpha, g_\beta, -1)}{\alpha g_\alpha + \beta g_\beta - g}. \tag{57}$$

In (57) the point $(g_\alpha, g_\beta, -1)$ fills an area of the plane $z = -1$ which folds back on itself along an edge, corresponding to the monoclastic curve $K = 0$, on which the Jacobian $\partial(g_\alpha, g_\beta)/\partial(\alpha, \beta)$ vanishes. Division by $\alpha g_\alpha + \beta g_\beta - g$ converts this edge into a more general edge of regression, the reason why it cannot open it up into a smooth surface being that the gradient of this denominator is parallel to the gradients of g_α and g_β—which are themselves parallel if $K = 0$, being tangent to the edge itself.

Another approach to this result uses a relation, which is in any case worth quoting, between the curvature K of S and the curvature $\bar{K}$ at the corresponding point of $\bar{S}$. This is

$$K\bar{K} = \frac{\sin^6 \phi}{N^2}, \tag{58}$$

where ϕ is the angle between the radius vector and the tangent plane (ϕ has the same value for a point on S and for the corresponding point on $\bar{S}$). From (58) it follows that, when $K = 0$, $\bar{K} = \infty$ (corresponding to an edge of regression), unless $\phi = 0$, when one of the points must be at infinity.

By either argument we see that surfaces of constant phase are cusped along edges, corresponding to any monoclastic curves of S. To obtain the amplitude variation in the direction of a cuspidal edge, we must modify the analysis leading to theorem 1, beginning at equation (35); for since $K = 0$, one of the principal curvatures, say κ_γ, is zero, and the more accurate approximation

$$\alpha = \alpha_m + \tfrac{1}{2}\kappa_\beta(\beta - \beta_m)^2 + \lambda_\gamma(\gamma - \gamma_m)^3 \tag{59}$$

to S is required. Using the equation

$$\int_{-\infty}^{\infty} \exp\,[\mathrm{i}h(\gamma - \gamma_m)^3]\,\mathrm{d}\gamma = \frac{(\frac{1}{3})!\sqrt{3}}{|h|^{\frac{1}{3}}}, \tag{60}$$

we deduce that

$$u_m = \pi \mathrm{i} \frac{F(\mathbf{k}_m)}{G_\alpha(\mathbf{k}_m)} \exp\,[\mathrm{i}\alpha_m x + \tfrac{1}{4}\pi\mathrm{i}\,\mathrm{sgn}\,\kappa_\beta] \left(\frac{2\pi}{x\,|\kappa_\beta|}\right)^{\frac{1}{2}} \frac{(\frac{1}{3})!\sqrt{3}}{(|x|\,\lambda_\gamma)^{\frac{1}{3}}}. \tag{61}$$

The most important point to notice is that the wave amplitude decays like $r^{-\frac{5}{6}}$ in the direction of the cuspidal edge, a rate of decay intermediate between those for cylindrical and spherical waves. To obtain a continuous variation between this result and the decay like r^{-1} which obtains on neighbouring radii one would have to retain both square and cube terms in $\gamma - \gamma_m$, leading to an answer involving the Airy integral. This analysis is omitted.

A final special case, which combines features of the discussions of both developable surfaces and monoclastic curves, is that in which, although S is not developable, normals to it lie in one particular direction l, not at a finite set of points (as assumed in theorem 1) but along a whole curve M. For example, this is so for a torus if l is its axis of symmetry. It may be shown that such a curve M is necessarily monoclastic, but instead of the corresponding curve on $\bar{S}$ being an edge of regression, the edge collapses into a single point, near which $\bar{S}$ has the shape of a double cone with it as vertex. The asymptotic contribution to u_0 on l from the curve M is analyzed by the method leading to (55), and the answer is the same with L replaced by M, γ by an arc-length along M and κ_β by the non-zero principal curvature on M. Therefore, l is an isolated direction in which the waves decay like $r^{-\frac{1}{2}}$ and the surfaces of constant phase have conical nodes.

6. Application of the radiation condition

It was pointed out at the beginning of § 3 that the solution of our differential equation (23) with given right-hand side is not unique, in that any complementary function (solution of the equation with zero right-hand side) can be added on to the particular integral so far studied. However, in any physical application only one solution is of interest, namely, that which satisfies the 'radiation condition'. This states that only waves originating at the source are present; no free waves are crossing the field from one side to the other, or simultaneously coming in from infinity in all directions. In this section, and in appendix B, different ways of applying the radiation condition, to derive the unique solution appropriate in such physical problems, and its asymptotic behaviour, will be discussed.

The simplest approach from a mathematical point of view is to replace ω by $\omega - \mathrm{i}\epsilon$, where $\epsilon > 0$, and afterwards to let $\epsilon \to 0$. The physical idea behind this is that the source strength is being taken as

$$\exp\left[\mathrm{i}(\omega-\mathrm{i}\epsilon)t\right] f(x,y,z), \tag{62}$$

which increases exponentially with time like $\mathrm{e}^{\epsilon t}$, while the solutions sought are also proportional to $\exp\left[\mathrm{i}(\omega-\mathrm{i}\epsilon)\,t\right]$, and so increase their amplitude in step with the source strength. Suppose now that the solution were contaminated by the presence of any waves 'coming in from infinity', and starting on their journey inward at time t_0, their amplitude being of order $\mathrm{e}^{\epsilon t_0}$, comparable with that of the waves generated at the source. Being free waves, these would not increase exponentially with time, so that on reaching the neighbourhood of the source they would still have amplitude of order $\mathrm{e}^{\epsilon t_0}$. But, by then, this would be negligible compared with $\mathrm{e}^{\epsilon t}$, the order of magnitude of waves generated at the source, because of the very large time needed for the free waves to come in from infinity. Hence, if we seek only the solution of order $\mathrm{e}^{\epsilon t}$, they must be absent.

Accordingly, with the source strength as in (62), this solution proportional to $\exp\left[\mathrm{i}(\omega-\mathrm{i}\epsilon)\,t\right]$ is the unique solution satisfying the radiation condition; and we may expect that its limit as $\epsilon \to 0$ is the unique solution of the original problem which satisfies that condition. This limit will now be determined, while, at the end of this section and in appendix B, different checks on the conclusion will be made.

The main alteration which occurs, when ω is replaced by $\omega - \mathrm{i}\epsilon$, is in the initial estimation of the inner integral in (31) for u_0 (where now $u = \exp\left[\mathrm{i}(\omega-\mathrm{i}\epsilon)\,t\right] u_0$). The denominator G is slightly changed by the said replacement, and its approximate new form is

$$G - \mathrm{i}\epsilon\,\partial G/\partial\omega, \tag{63}$$

if G stands for the original G (with $\epsilon = 0$) and $\partial G/\partial\omega$ refers to the fact that equation (27) makes it a function of ω as well as of α, β and γ.

Accordingly, the zeros of the denominator are slightly changed. Instead of being those real values of α where $G = 0$, they are displaced to slightly different, and in general complex, positions. Near a real simple zero $\alpha = \alpha_0$ of G, the denominator (63) is approximately

$$(\alpha-\alpha_0)\frac{\partial G}{\partial\alpha} - \mathrm{i}\epsilon\frac{\partial G}{\partial\omega}, \tag{64}$$

which vanishes at

$$\alpha = \alpha_0 + \mathrm{i}\epsilon\frac{\partial G/\partial\omega}{\partial G/\partial\alpha}, \tag{65}$$

so that the displacement is into the upper half plane if

$$\frac{\partial G/\partial \omega}{\partial G/\partial \alpha} > 0, \tag{66}$$

and into the lower half plane if the opposite inequality holds.

Now, the inner integral in (31) can be asymptotically estimated as $x \to +\infty$ by shifting the path of integration to the line $\mathfrak{J}\alpha = h > 0$ (reducing the order of magnitude of the integral to e^{-hx}, which will be neglected) and adding on $2\pi i$ times the sum of the residue of the integrand at all poles with $0 < \mathfrak{J}\alpha < h$. For suitably chosen h and small enough ϵ, these poles are at those values of (65) for which the inequality (66) holds. The residues are approximately F/G_α for small ϵ. Hence in place of (32) we have

$$2\pi i \sum_{S_+} \frac{F(\alpha, \beta, \gamma)}{G_\alpha(\alpha, \beta, \gamma)} e^{i\alpha x} + O(x^{-N}), \tag{67}$$

where $\sum_{S_+}$ is a sum over those values of α such that (α, β, γ) is on S_+, the part of S where (66) holds. Thus, the contribution of those zeros of G where (66) holds has been doubled, and that of the others annulled.

Accordingly, equation (33) is replaced by

$$u_0 \sim 2\pi i \iint_{S_+} \frac{F(\alpha, \beta, \gamma)}{G_\alpha(\alpha, \beta, \gamma)} e^{i\alpha x} \, d\beta \, d\gamma. \tag{68}$$

The estimation of this integral now follows exactly the course taken in § 3 for the integral (33), leading to an expression as a sum of terms (38), but doubled and restricted to those $\mathbf{k}_m$ which are on S_+. They are next put into a form invariant under rotation of axes. At the same time, the condition (66) must be put into such a form, as

$$\frac{\mathbf{r} \, . \, \nabla G}{\partial G/\partial \omega} > 0, \tag{69}$$

where the fraction has been inverted for convenience later on. We can now state theorem 2 below.

Before doing so, however, we note one point in the argument which (like the corresponding point in the proof of theorem 1) might give pause to some readers. This is the possibility of an asymptotic contribution to (68) from curves on S_+ where $G_\alpha = 0$ (so that the x-direction is tangential to S instead of normal), which for non-zero G_ω (as obtains in practice for reasons to be given below) would be *boundary* curves of S_+. A special investigation, studying among others the contribution from a line $\beta, \gamma =$ const. which just fails to touch the surface at such a curve, is necessary to satisfy oneself that the sum of all such contributions is zero.

As in § 3, however, this is more obvious from an approach in which the integrations with respect to β and γ are carried out first. The function

$$V(\alpha) = \lim_{\epsilon \to 0} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{F(\alpha, \beta, \gamma)}{G(\alpha, \beta, \gamma, \omega - i\epsilon)} \, d\beta \, d\gamma \tag{70}$$

can be shown to have singularities only for values of α such that the plane $\alpha =$ const. touches the surface $G = 0$. The singularity of $V(\alpha)$ at $\alpha = \alpha_m$ is like

$$\frac{F(\mathbf{k}_m)}{G_\alpha(\mathbf{k}_m)} \frac{2\pi i}{\sqrt{|K|}} \{\mp \log |\alpha - \alpha_m| + \tfrac{1}{2}\pi i \, \mathrm{sgn}\,(\alpha - \alpha_m)\} \tag{71}$$

if $K < 0$, and like

$$\frac{F(\mathbf{k}_m)}{G_\alpha(\mathbf{k}_m)} \frac{2\pi}{\surd K} \operatorname{sgn}(\kappa_\beta + \kappa_\gamma) \{\log|\alpha - \alpha_m| \pm \tfrac{1}{2}\pi \mathrm{i} \operatorname{sgn}(\alpha - \alpha_m)\} \tag{72}$$

if $K > 0$, the upper sign being taken in each if (66) holds and otherwise the lower. (In the parallel investigation which was noted in § 3, the terms of variable sign in (71) and (72) are simply absent.) From table 1 of Lighthill (1958), we can now determine the asymptotic form as $x \to +\infty$ of the Fourier transform

$$u_0 = \int_{-\infty}^{\infty} V(\alpha)\, \mathrm{e}^{\mathrm{i}\alpha x}\, \mathrm{d}\alpha; \tag{73}$$

it comes out as (38) multiplied by 2 if the upper sign has been taken, by 0 if the lower has been taken, and by 1 if the terms of variable sign are suppressed. This checks the previous conclusions; and, in particular, we have

THEOREM 2. *The solution of*

$$P\left(\frac{\partial^2}{\partial t^2}, \frac{\partial^2}{\partial x^2}, \frac{\partial^2}{\partial y^2}, \frac{\partial^2}{\partial z^2}\right) u = \mathrm{e}^{\mathrm{i}\omega t} f(x, y, z), \tag{23 ter}$$

which satisfies the radiation condition, is asymptotically

$$u = \frac{4\pi^2\, \mathrm{e}^{\mathrm{i}\omega t}}{r} \Sigma \frac{CF \exp[\mathrm{i}(\alpha x + \beta y + \gamma z)]}{|\nabla G| \surd|K|} + O\left(\frac{1}{r^2}\right), \tag{74}$$

as $r \to \infty$ along any radius vector l, if the sum Σ is over all points (α, β, γ) of the surface $G = 0$ where the normal to the surface is parallel to l and $(\mathbf{r}\,.\,\nabla G)/(\partial G/\partial \omega) > 0$; provided that the surface has non-zero Gaussian curvature K at each of these points; that C is (i) $\pm \mathrm{i}$ where $K < 0$ and ∇G is in the direction of $\pm r$, (ii) ± 1 where $K > 0$ and the surface is convex to the direction of $\pm \nabla G$; that

$$F(\alpha, \beta, \gamma) = \frac{1}{8\pi^3} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y, z) \exp[-\mathrm{i}(\alpha x + \beta y + \gamma z)]\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z; \tag{42 bis}$$

and that

$$G(\alpha, \beta, \gamma, \omega) = P(-\omega^2, -\alpha^2, -\beta^2, -\gamma^2). \tag{75}$$

The restriction (69), to which we are led by this mathematically simple method of applying the radiation condition, has a direct physical interpretation. In any plane wave

$$u = a \exp[\mathrm{i}(\omega t + \alpha x + \beta y + \gamma z)] \tag{76}$$

travelling through a conservative system, for which the laws of mechanics lead to an equation $G(\alpha, \beta, \gamma, \omega) = 0$ relating frequency and wave number, the velocity of energy propagation (group velocity) can be determined as

$$\mathbf{U} = \frac{\nabla G}{\partial G/\partial \omega}. \tag{77}$$

The most general approach to proving this result is an extension to three dimensions of the energy argument of Rayleigh (1894, p. 479); this does not seem to have been given before and therefore is set out in appendix A.

Accordingly, the restriction (69) is simply $\mathbf{r}\,.\,\mathbf{U} > 0$, which has the clear physical interpretation that the only waves occurring are those for which the velocity of energy propagation has a positive (outward) component along a radius vector. In other words, all the energy is created at the source; none comes in from infinity. This gives a check on the mathematical method by making physical sense out of the answer.

Indeed, equation (77) makes physical sense out of other aspects of theorem 2, which states that $\mathbf{r}$ is actually parallel to ∇G, and hence to $\mathbf{U}$; thus, energy spreads outwards from a source directly along radii. However tortuous the surfaces of constant phase, the energy follows a straight line path.

Finally, equation (77) justifies us for having ignored, when proving theorem 2, the possibility of critical cases when $\partial G/\partial\omega = 0$ (rendering indeterminate whether the pole of G as function of α, given to a first approximation by (64), is above, below, or on the real axis), since these would correspond physically to an infinite velocity of energy propagation, which is impossible.

We end this section with notes on the special case of non-dispersive waves. These are waves governed by an equation (23) in which the polynomial P is homogeneous in its four variables, as in the case of equation (17). Then the phase velocity ω/k is independent of wavelength for plane waves travelling in a fixed direction.

If P is homogeneous of degree n, then G is homogeneous of degree $2n$, and by Euler's equation for homogeneous functions

$$\omega\,\partial G/\partial\omega+\mathbf{k}\,.\,\nabla G = 2nG = 0 \tag{78}$$

throughout the wave-number surface $G = 0$. Hence (69) requires that $\mathbf{k}\,.\,\nabla G$ and $\mathbf{r}\,.\,\nabla G$ are of opposite signs. Since $\mathbf{r}$ and ∇G are parallel, we conclude that for non-dispersive waves the restriction in theorem 2 takes the simple form

$$\mathbf{k}\,.\,\mathbf{r} < 0. \tag{79}$$

In this case, then, only waves whose phase velocity has an outward radial component are present.

However, phase velocity and group velocity must not be confused, even for non-dispersive waves, in an anisotropic medium. The group velocity (77) can by (78) be written

$$\mathbf{U} = -\omega\nabla G/\mathbf{k}\,.\,\nabla G, \tag{80}$$

while the phase velocity $\mathbf{c}$, written as a vector, is

$$\mathbf{c} = -\omega\mathbf{k}/k^2. \tag{81}$$

From (80) and (81) we see that $\mathbf{U}\,.\,\mathbf{k} = \mathbf{c}\,.\,\mathbf{k}$; thus the resultant of $\mathbf{U}$ in the direction of $\mathbf{c}$ is $\mathbf{c}$. In other words, the speed of energy propagation normal to the wave fronts is the phase velocity. But $\mathbf{U}$ has in general a component normal to $\mathbf{c}$; thus, in a plane wave, energy may be propagated parallel to the wave fronts. Indeed, whenever the wave velocity varies with direction this is so, and it can easily be shown that the speed of energy propagation along a line in the wave front is minus the rate of change of wave speed with angle as the direction of propagation sweeps along that line.

To check this result, Mr D. R. Bland has kindly investigated the case of a plane elastic wave in a general anisotropic solid, calculated the energy propagation velocity from the product of stress tensor and velocity vector, and obtained a value in agreement with (80).

This fact, that plane waves are possible only if there is an energy supply transmitting energy parallel to the wave fronts, re-emphasizes how unrealistic is a treatment of anisotropic wave motions in terms of plane waves alone.

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7. Effect of compressibility on magneto-hydrodynamic waves generated at a source

We now apply the mathematical theory of the last four sections to the partial differential equation (17), which is satisfied by Γ, the rate of strain along magnetic lines of force, in the theory of small disturbances to a compressible, perfectly conducting fluid in a uniform magnetic field. Possible right-hand sides to the equation, which can represent sources of magneto-hydrodynamic waves, were noted in equations (18) to (22).

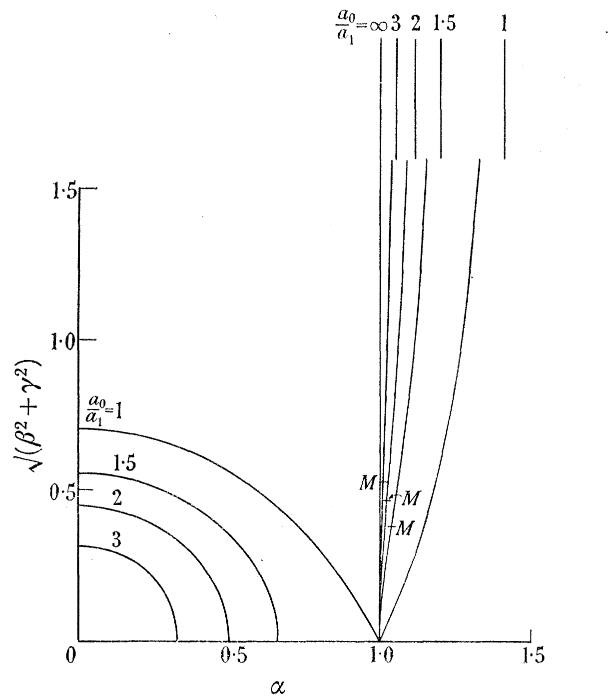

Figure 3. Compressibility effect on the shape of S. For $a_0/a_1 = 1$, 1·5, 2, 3 and ∞, S is obtained by rotating about the α-axis these curves together with their reflexions in the plane $\alpha = 0$. At the top of the figure the asymptotes of the curves are shown. The unit of wave number is ω/a_1. The points of inflexion, marked M, become monoclastic curves on S after rotation.

The theory gives results in terms of the geometry of the surface S whose equation is $G = 0$, where by (17), (23) and (27)

$$G = \omega^2\{\omega^2 - (a_0^2 + a_1^2)\, k^2\} + a_0^2 a_1^2 \alpha^2 k^2. \tag{82}$$

Solving $G = 0$ for $\beta^2 + \gamma^2$ (that is, $k^2 - \alpha^2$), as

$$\beta^2 + \gamma^2 = \frac{(\omega^2 - a_0^2\alpha^2)\,(\omega^2 - a_1^2\alpha^2)}{(a_0^2 + a_1^2)\,\omega^2 - a_0^2 a_1^2 \alpha^2}, \tag{83}$$

we see that S is a surface of revolution, with axis the α-axis and a plane of symmetry $\alpha = 0$; it is in three sheets, one an ovoid of major axis ω/a_0 and minor axis $\omega/\surd(a_0^2 + a_1^2)$, and the other two consisting of the planes $\alpha = \pm\omega\surd(a_0^{-2} + a_1^{-2})$ distorted by bumps near the x-axis, which reduce the distance between the distorted surfaces to twice ω/a_1. (In this description it has been assumed that $a_1 < a_0$; otherwise the description is still correct if in it a_0 and a_1 are interchanged.) Figure 3 illustrates the shape of S for the values 1, 1·5, 2, 3 and ∞ of the ratio

a_0/a_1, taking ω/a_1 as the unit of wave number in each case. Note that the values 1, $\frac{2}{3}$, $\frac{1}{2}$, $\frac{1}{3}$ and 0 of the ratio a_0/a_1 are also covered by figure 3 if ω/a_0 be taken as the unit of wave number.

The limit $a_0/a_1 = \infty$ is the incompressible case, when S degenerates into the two planes $\alpha = \pm\omega/a_1$ and the ovoid collapses to a point. This shape for S, of two planes perpendicular to the x-direction, corresponds according to the theory of § 5 to the physical property (§ 1) of magneto-hydrodynamic waves in an incompressible, perfectly conducting fluid, that the motion propagates only in the x-direction (normal to the said planes) and without attenuation.

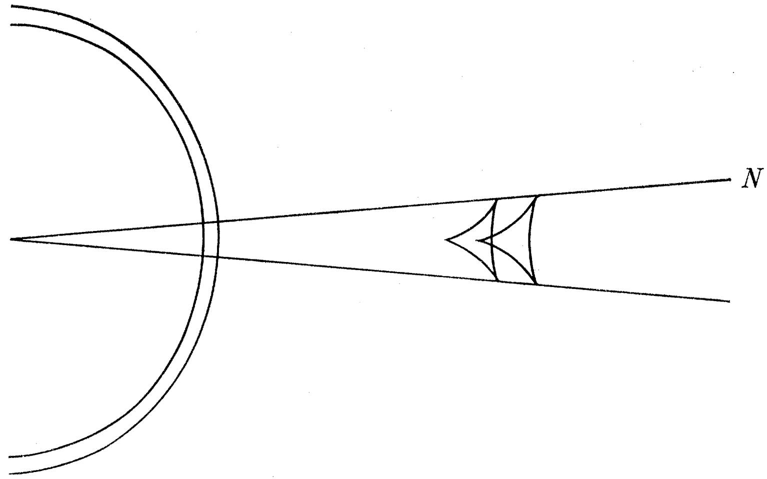

FIGURE 4. Compressibility effect on surfaces of constant phase. For $a_0/a_1 = 2$ this shows two surfaces of constant phase generated by the ovoid sheet of S and two generated by the distorted-plane sheet (only the part with $x > 0$ in each case). The phases, from left to right, are in the ratios 1:1·08:5:5·4.

For finite values of the ratio a_0/a_1, the planes form bumps as noted above, so that one-dimensional propagation of values of Γ ceases, although it was shown in § 2 that one-dimensional propagation of values of ξ, the vorticity component along magnetic lines of force, is retained. Waves whose wave-number vectors lie on the distorted-plane sheets of S are present along each radius parallel to the normal at some point of those sheets. Such radii lie within a right circular cone N, whose generators are parallel to the normals at points of the monoclastic curve M, a circle which intersects a given meridian plane (for example, figure 3) at the point of inflexion, marked M on each curve.

From the theory of § 5, the surfaces of constant phase within this cone (which are reciprocal polars of the distorted-plane sheets of S) have cuspidal edges corresponding to the monoclastic curve M, and therefore lying on the cone N. Figure 4 shows these surfaces in the case $a_0/a_1 = 2$, when the semi-angle of the cone is 4·7°. For higher values of the ratio a_0/a_1 one finds that the semi-angle to a close approximation is $(\frac{3}{16}\sqrt{3})\,(a_1/a_0)^2$ radians, or $18{\cdot}6(a_1/a_0)^2$ degrees.

A further effect of a_0/a_1 being finite is that a new system of waves, associated with the ovoid branch of S, appear, and these can obviously be identified with sound waves. The associated surfaces of constant phase are also drawn in figure 4, for $a_0/a_1 = 2$; their distortion from the spherical shape is due to the magnetic restoring force which acts upon transverse movements, and increases the sound speed from a_0 along magnetic lines of force to $\sqrt{(a_0^2+a_1^2)}$ at right angles to them.

We may ask at this stage: what governs whether a given type of source generates principally sound waves or magneto-hydrodynamic waves? The answer depends on the presence of the F factor in the result of theorem 2, F being the Fourier transform of the source strength. The waves produced are mainly those associated with parts of S where F is not small. Thus, Fourier components in the source strength with small wave number tend to generate sound waves, and those with large wave number magneto-hydrodynamic waves. For an extended source of dimension l, F becomes small as kl becomes large (see the $\exp(-\frac{1}{4}k^2l^2)$ terms in (29)), so that the outer parts of the distorted-plane sheets of S do not contribute, and therefore the parts of the constant-phase surfaces of figure 4 which make a small angle with the axis must be absent.

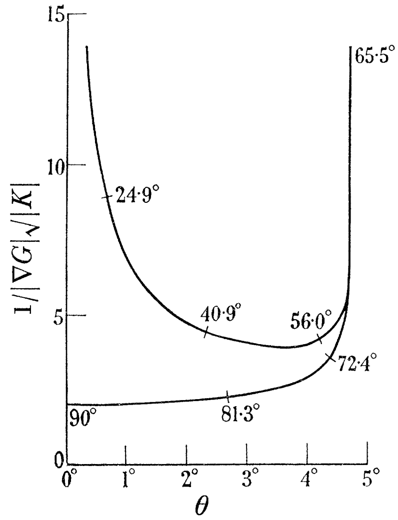

FIGURE 5. Variation of the amplitude factor $1/|\nabla G|\surd|K|$ with direction θ for $a_0/a_1 = 2$. The numbers on the curves are values of ϕ, the angle between wave crest and x-axis.

Equations (21) and (22) show how the distributions of Γ and of $\Delta = \text{div}\,\mathbf{v}$ may differ. In this case, of a motion generated by a distribution of fluctuating mass-sources, the values of F for the propagation of Γ and Δ, respectively, are given by the last two expressions in (29). In the case of Δ, F is especially small on the distorted-plane sheets of S, since $\alpha \doteqdot \omega/a_1$ on these. Thus, magneto-hydrodynamic waves for small a_1/a_0 are practically equivoluminal, Γ being much greater than Δ.

The volume of sound generated in this case will depend on the value of l. The maximum of $k^2 \exp[-\frac{1}{4}k^2l^2]$ is at $k = 2/l$, so that, if this is much greater than ω/a_0, and especially if it exceeds ω/a_1, less sound than magneto-hydrodynamic waves will be generated. If doublets instead of sources were the producers of sound, an additional factor $\mathbf{k}.\mathbf{n}$ ($\mathbf{n}$ being the doublet direction) would be present in (29), which would accentuate the effect.

Another difference in intensity between sound waves and magneto-hydrodynamic waves arises from the $1/\surd|K|$ factor in (74), which is obviously greater on the distorted-plane sheets of S than on the ovoid sheet. This factor increasing the relative intensity of magneto-hydrodynamic waves can be regarded, according to arguments given at the end of § 4, as due to their being concentrated in a narrow cone, while sound waves can freely spread in all directions.

The distribution of wave amplitude with direction inside this narrow cone can be inferred from figure 5, which plots $1/|\nabla G|\surd|K|$ against $\theta = \cos^{-1}(x/r)$, the angle between radius vector and x-axis. Values of $\phi = \sin^{-1}(\alpha/\kappa)$, the angle between the wave crests and the x-axis, are noted on the curve. The ordinate needs to be multiplied by $(4\pi^2/r)\,F$, a factor depending, as discussed above, on the details of the source, to give the wave amplitude. For any source of non-zero length scale, the rapid decay of F as $k \to \infty$ will bring the upper branch of figure 5 down to zero (instead of increasing to infinity) as $\theta \to 0$. Also, by § 5, the infinite amplitude on the conical boundary, where the waves have cusps, is not exact; there is really decay like $r^{-\frac{5}{6}}$ instead of like r^{-1} along this boundary.

It has been seen that a fairly precise measure of the degree of departure of magneto-hydrodynamic waves from strictly one-dimensional propagation, as a result of compressibility, can be given by means of the mathematical theory which has been set out. We now consider modifications due to another effect neglected in the simple theory.

8. Hall-current term in the equations of motion

The situations, mainly astrophysical, in which dissipative effects are small enough to permit the propagation of magneto-hydrodynamic waves without rapid attenuation, involve a high-temperature, low-density plasma of electrons and positive ions. We investigate now whether equation (7), used in § 2 for the rate of change of magnetic induction, is a good approximation in such a plasma.

This equation stems from the basic Maxwell law of induction,

$$\partial \mathbf{B}/\partial t = -\operatorname{curl} \mathbf{E}, \tag{84}$$

and the idea that the Lorentz force on a particle moving with velocity $\mathbf{v}$, namely $\mathbf{E}+\mathbf{v}\wedge\mathbf{B}$, must vanish in a perfectly conducting fluid, because charge displacements such as reduce it to zero take a negligible time.

The difficulty with this argument is that $\mathbf{v}$ is not the velocity, or even the mean velocity, of the particles responsible for charge displacements, namely, the electrons. This is because, for the hydrodynamical equations (4) and (5) to hold, $\mathbf{v}$ must be the momentum of the gas per unit volume, divided by its density. Hence, the mass m_i of the ions being over 1800 times the electronic mass m_e, $\mathbf{v}$ must be far closer to $\mathbf{v}_i$, the mean velocity of the ions, than to $\mathbf{v}_e$, that of the electrons.

In fact, if n_i and n_e are the number densities of ions and electrons, we have

$$\rho = n_i m_i + n_e m_e, \quad \mathbf{v} = \frac{n_i m_i \mathbf{v}_i + n_e m_e \mathbf{v}_e}{n_i m_i + n_e m_e}. \tag{85}$$

Also, if Z is the average charge of the ions, we have to a very close approximation $n_e = n_i Z$ (electrical neutrality); for, after any calculation of the electric field $\mathbf{E}$ (such as we are engaged in), the net charge density can be determined as $(\operatorname{div}\mathbf{E})/4\pi c^2$ in e.m.u., which is always very small indeed compared with the electron charge density $(-en_e)$ because of the c^2 in the denominator. It follows that $(n_e m_e)/(n_i m_i) \doteqdot Z m_e/m_i < \frac{1}{1800}$, so that (85) places $\mathbf{v}$ extremely close to $\mathbf{v}_i$. Furthermore, the current density $\mathbf{j}$ is

$$\mathbf{j} = e(n_i Z \mathbf{v}_i - n_e \mathbf{v}_e) \doteqdot en_e(\mathbf{v}_i - \mathbf{v}_e) \doteqdot en_e(\mathbf{v} - \mathbf{v}_e), \tag{86}$$

and therefore the mean velocity of the electrons is approximately

$$\mathbf{v}_e = \mathbf{v} - \mathbf{j}/en_e. \tag{87}$$

Of course, the spread of electron velocities about their mean $\mathbf{v}_e$ greatly exceeds that of the ion velocities about $\mathbf{v}_i$, the ratio of standard deviations being for example, $(m_i/m_e)^{\frac{1}{2}}$, in thermodynamic equilibrium. Thus, the pressure p is made up of an electron pressure p_e and an ion pressure p_i which are of the same order of magnitude, being in the ratio $n_e/n_i = Z$ in thermodynamic equilibrium.

We return now to the problem of determining the rate of change of the magnetic induction $\mathbf{B}$, given the existing value of $\mathbf{B}$, and hence that of the current density,

$$\mathbf{j} = (\operatorname{curl} \mathbf{B})/4\pi. \tag{88}$$

The idea that $\mathbf{E}+\mathbf{v}\wedge\mathbf{B}$ vanishes has already been criticized; this is the mean Lorentz force on the ions, which in practice will be balanced against their rate of change of momentum. However, one might expect the rate of change of electron momentum to be far smaller, in which case it would be more accurate to put $\mathbf{E}+\mathbf{v}_e\wedge\mathbf{B} = 0$.

The true situation is a little more complicated, because the rate of change of electron momentum per unit volume includes terms due to the large random fluctuations of electron velocity about their mean $\mathbf{v}_e$. By the definition of pressure in terms of momentum flux, these can be written as the gradient of electron pressure, ∇p_e. Therefore, if we are willing to neglect the electron-inertia effects, and the momentum exchange between electrons and ions (the electrical-resistivity term), we can balance ∇p_e against the Lorentz force per unit volume on the electrons, $-n_e e(\mathbf{E}+\mathbf{v}_e\wedge\mathbf{B})$, to give

$$\frac{\partial \mathbf{B}}{\partial t} = -\operatorname{curl} \mathbf{E} = \operatorname{curl}\left(\mathbf{v}_e\wedge\mathbf{B}+\frac{\nabla p_e}{n_e e}\right) = \operatorname{curl}(\mathbf{v}_e\wedge\mathbf{B}). \tag{89}$$

Here, the curl of $(\nabla p_e)/n_e$ has been put equal to zero on the ground that the electron pressure p_e will fluctuate in a direct functional relationship to the density and so also to n_e.

The standard equation (7) has the well-known interpretation that the magnetic lines of force 'move with', or 'are frozen into', the gas. The more accurate equation (89) means that they move with, or are frozen into, the *electron* gas.

To put the argument leading to (89) more formally, we may write down the momentum equations for the electrons and ions separately, as

$$n_e m_e\left(\frac{\partial \mathbf{v}_e}{\partial t}+\mathbf{v}_e.\,\nabla\mathbf{v}_e\right) = -\nabla p_e-\mathbf{M}-n_e e(\mathbf{E}+\mathbf{v}_e\wedge\mathbf{B}), \tag{90}$$

$$n_i m_i\left(\frac{\partial \mathbf{v}_i}{\partial t}+\mathbf{v}_i.\,\nabla\mathbf{v}_i\right) = -\nabla p_i+\mathbf{M}+n_i Ze(\mathbf{E}+\mathbf{v}_i\wedge\mathbf{B}). \tag{91}$$

Here, $\mathbf{M}$ is the rate of loss of electron momentum, per unit volume, by collisions with ions; it may be supposed proportional to their mean relative velocity $\mathbf{v}_e-\mathbf{v}_i$, and therefore to $(-\mathbf{j})$; and, indeed, if we write

$$\mathbf{M} = -n_e e\eta\mathbf{j}, \tag{92}$$

then η may be identified with the resistivity, in that (90) becomes Ohm's law $\mathbf{E} = \eta\mathbf{j}$ under uniform conditions with $\mathbf{B} = 0$.

Note that the hydrodynamical momentum equation (5) is the sum of (90) and (91). This is obvious as far as the right-hand sides are concerned if we accept that $n_e = n_i Z$. The left-hand sides of (90) and (91) do not exactly add up to that of (5), although to a close approximation those of (5) and (91) are equal and that of (90) negligible; the exact difference is physically the effect of the pressure of the current flow, namely, $m_e j^2/e^2 n_e$ along lines of current flow. Such a term of order j^2 is normally neglected, and certainly can be in a theory of first-order perturbations to a state of zero current.

To obtain $\mathbf{E}$, and hence $\partial\mathbf{B}/\partial t$, we now argue that, because the left-hand side of (90) is very small compared with that of (91), while individual terms on their right-hand sides are of the same order of magnitude, we can get a good approximation by putting that of (90) equal to 0, giving

$$\mathbf{E}+\mathbf{v}_e \wedge \mathbf{B} = \eta\mathbf{j}-(\nabla p_e)/n_e e. \tag{93}$$

Under stationary, uniform conditions this becomes

$$\mathbf{E} = \eta\mathbf{j}+\frac{\mathbf{j}\wedge\mathbf{B}}{n_e e}, \tag{94}$$

by (87) with $\mathbf{v} = 0$. This well-known equation for the current $\mathbf{j}$ has solutions which consist of the sum of the 'ohmic current' $\mathbf{E}/\eta$ and a so-called 'Hall current' perpendicular to $\mathbf{B}$. The latter arises from the term in (94) owing to our substitution of $\mathbf{v}_e$ for $\mathbf{v}$, which can accordingly be described as the Hall-current effect.

The relative importance of the two terms in (94) depends on the magnitude of the ratio

$$B/n_e e\eta. \tag{95}$$

When this ratio is large, the Hall-current term is more important than the resistivity term. The ratio can also be written ω_e/ω_c, where

$$\omega_e = Be/m_e, \quad \omega_c = n_e e^2\eta/m_e; \tag{96}$$

here, ω_e is the gyro-frequency of the electrons (rate of spiralling about magnetic lines of force), and ω_c is an average frequency of electron collisions with ions; for, by (86) and (92),

$$\mathbf{M} = n_e^2 e^2 \eta(\mathbf{v}_e-\mathbf{v}_i) = \omega_c m_e n_e(\mathbf{v}_e-\mathbf{v}_i), \tag{97}$$

so that the rate of momentum loss of electrons per unit volume, by collision with ions, is equal to ω_c times the mean value of their momentum in a frame in which the mean ion velocity is zero.

Thus, the Hall-current effect is much more important than the electrical resistivity whenever the magnetic field B is so large that the gyro-frequency of electrons greatly exceeds their collision frequency. Taking $\eta \simeq 10^{14} T^{-\frac{3}{2}}$ (Spitzer 1956) in (95), this requires

$$B \gg 10^{-6} n_e T^{-\frac{3}{2}} \tag{98}$$

(B in gauss, n_e in cm^{-3}, T in °K), so that conditions of high temperature and low density are precisely those in which the condition can be satisfied for reasonable field strengths.

In this case it may be of value to study the non-dissipative form of (93), with the resistivity omitted—which gives equation (89). Substituting for $\mathbf{v}_e$ in this from (87) and (88), we get

$$\frac{\partial\mathbf{B}}{\partial t} = \operatorname{curl}\left\{\left(\mathbf{v}-\frac{\operatorname{curl}\mathbf{B}}{4\pi n_e e}\right)\wedge\mathbf{B}\right\}, \tag{99}$$

an equation for $\mathbf{B}$ which, with (4), (5) and (6) for ρ, $\mathbf{v}$ and p, can be solved by steps forward in time.

We conclude this section by referring briefly to the effects besides resistivity which have been neglected in reaching equation (99). These effects, mainly electron inertia and departures from electrical neutrality, are known to be unimportant (Spitzer 1956) if the frequency ω is small compared with the 'plasma oscillation frequency',

$$\omega_p = \left(\frac{4\pi n_e}{m_e}\right)^{\frac{1}{2}} ec = (6\times 10^4 n_e^{\frac{1}{2}})\ \mathrm{sec}^{-1}. \tag{100}$$

9. Hall-current effect on magneto-hydrodynamic waves

In §§2 and 7 we investigated how far compressibility interferes with the unidirectional propagation of magneto-hydrodynamic waves, finding that it does not as far as the vorticity component ξ is concerned. We ask now the same question about the Hall-current effect, but in order to keep the analysis tractable we limit it to an incompressible fluid, particularly as the effect of compressibility on its own has already been evaluated.

Accordingly, we use equation (99) for **B**, and equation (5) for **v**, but replace equation (6) by $\rho = \text{const.} = \rho_0$, so that (4) becomes

$$\operatorname{div}\mathbf{v} = 0. \tag{101}$$

Then, when departures of **B** and **v** from uniform values $\mathbf{B}_0$ and 0 are regarded as so small that their squares and products are negligible, equations (99) and (5) become

$$\frac{\partial \mathbf{B}}{\partial t} = \operatorname{curl}\left\{\left(\mathbf{v}-\frac{\operatorname{curl}\mathbf{B}}{4\pi n_e e}\right)\wedge \mathbf{B}_0\right\} \tag{102}$$

and
$$\rho_0\frac{\partial \mathbf{v}}{\partial t} = -\nabla p + \frac{(\operatorname{curl}\mathbf{B})\wedge\mathbf{B}_0}{4\pi} = -\nabla\left(p+\frac{\mathbf{B}_0\,.\,\mathbf{B}}{4\pi}\right)+\frac{\mathbf{B}_0\,.\,\nabla\mathbf{B}}{4\pi}. \tag{103}$$

Now, taking the divergence of equation (103), we obtain

$$\nabla^2\left(p+\frac{\mathbf{B}_0\,.\,\mathbf{B}}{4\pi}\right) = 0, \tag{104}$$

of which the only solution bounded throughout space is

$$p+\frac{\mathbf{B}_0\,.\,\mathbf{B}}{4\pi} = \text{const.} \tag{105}$$

Equation (105) justifies the remark in §1, that magnetic pressure variations are exactly balanced by those of the gas pressure for an incompressible fluid.

Accordingly, in axes such that $\mathbf{B}_0 = (B_0, 0, 0)$, equations (102) and (103) give

$$\frac{\partial \mathbf{B}}{\partial t} = B_0\frac{\partial}{\partial x}\left(\mathbf{v}-\frac{\operatorname{curl}\mathbf{B}}{4\pi n_e e}\right),\quad \frac{\partial \mathbf{v}}{\partial t} = \frac{B_0}{4\pi\rho_0}\frac{\partial \mathbf{B}}{\partial x}, \tag{106}$$

whence
$$\frac{\partial}{\partial t}\left(\frac{\partial}{\partial t}+\frac{B_0}{4\pi n_e e}\frac{\partial}{\partial x}\operatorname{curl}\right)\mathbf{v} = \frac{B_0^2}{4\pi\rho_0}\frac{\partial^2 \mathbf{v}}{\partial x^2}. \tag{107}$$

The coefficient $B_0^2/4\pi\rho_0$ in (107) is a_1^2, whilst

$$\frac{B_0}{4\pi n_e e} = \frac{a_1^2\rho_0}{B_0 n_e e} = \frac{a_1^2 m_i n_i}{B_0 n_e e} = \frac{a_1^2}{\omega_i}, \tag{108}$$

where
$$\omega_i = \frac{B_0(Ze)}{m_i}. \tag{109}$$

is the gyro-frequency of the ions (which is over 1800 times less than the gyro-frequency ω_e defined in (96)).

Equation (107) has three dependent variables v_x, v_y and v_z, but these are linked also by equation (101). To get convenient equations in only two variables we may use ξ and Γ again, since these (§ 2) determine $\mathbf{v}$ completely when $\Delta = 0$. The curl of (107), and the derivative of its x-component with respect to x, become, after the substitution (108),

$$\frac{\partial^2\xi}{\partial t^2}-\frac{a_1^2}{\omega_i}\frac{\partial}{\partial t}\nabla^2\Gamma = a_1^2\frac{\partial^2\xi}{\partial x^2},\quad \frac{\partial^2\Gamma}{\partial t^2}+\frac{a_1^2}{\omega_i}\frac{\partial^3\xi}{\partial t\,\partial x^2} = a_1^2\frac{\partial^2\Gamma}{\partial x^2}, \tag{110}$$

whence ξ itself satisfies

$$\left\{\left(\frac{\partial^2}{\partial t^2}-a_1^2\frac{\partial^2}{\partial x^2}\right)^2+\frac{a_1^4}{\omega_i^2}\frac{\partial^2}{\partial t^2}\frac{\partial^2}{\partial x^2}\nabla^2\right\}\xi = 0, \tag{111}$$

and Γ satisfies the same equation.

Equation (111) shows that the Hall-current effect prevents even the vorticity component ξ from being propagated one-dimensionally. The actual manner in which it is propagated can be found, according to the theory of §§ 3 to 6, in terms of the geometry of the surface S whose equation is $G = 0$, where by (111), (23) and (27)

$$G = (\omega^2-a_1^2\alpha^2)^2-a_1^4\alpha^2k^2(\omega^2/\omega_i^2). \tag{112}$$

Solving $G = 0$ for $\beta^2+\gamma^2$ (that is, $k^2-\alpha^2$), as

$$\beta^2+\gamma^2 = \frac{\omega_i^2}{\omega^2}\frac{1}{\alpha^2}\left(\frac{\omega^2}{a_1^2}-\alpha^2\right)^2-\alpha^2, \tag{113}$$

we see that S is a surface of revolution, with axis the α-axis and a plane of symmetry $\alpha = 0$; for $\omega < \omega_i$, it has two sheets S_1 and S_2 in $\alpha > 0$, lying respectively in the regions

$$0 < \alpha \leqslant \frac{\omega}{a_1}\left(1+\frac{\omega}{\omega_i}\right)^{-\frac{1}{2}} \quad\text{and}\quad \alpha \geqslant \frac{\omega}{a_1}\left(1-\frac{\omega}{\omega_i}\right)^{-\frac{1}{2}}, \tag{114}$$

and two symmetrically placed sheets in $\alpha < 0$. When, however, $\omega > \omega_i$, the sheet S_2 disappears and only S_1 is present. Figure 6 illustrates the shape of S in $\alpha > 0$ for the values 0, $\frac{1}{4}$, $\frac{1}{2}$, $\frac{3}{4}$, 1, $\frac{3}{2}$, 3 and 10 of the ratio ω/ω_i.

We see that the frequency ω_i is of critical importance for magneto-hydrodynamic waves, as was found already by Åström (1951). Note that waves with frequencies in this neighbourhood can justifiably be treated by the equations of § 8, because ω_i is always very small compared with the plasma oscillation frequency ω_p of equation (100), provided only that

$$B_0 \ll (6n_e^{\frac{1}{2}})\ \text{gauss}. \tag{115}$$

This condition is amply satisfied in all conditions for which the waves have been discussed, and is fully compatible with (98).

The limit $\omega/\omega_i = 0$, when S_1 and S_2 both become the plane $\alpha = \omega/a_1$, is the case of the 'perfectly conducting fluid', transmitting waves one-dimensionally in the x-direction. As ω/ω_i increases from zero, we see that the plane splits into two surfaces S_1 and S_2, so that the single system of waves splits into two separate systems. It is the appearance of coupling terms in the equations (110) for ξ and Γ which effects this splitting.

The two systems of waves are illustrated in figure 7 for a fairly small value of ω/ω_i, namely, $\frac{1}{4}$. The waves corresponding to S_1 lie within a cone N_1 of semi-angle 4·8°, on which the surfaces of constant phase have cusps, the situation being analogous to one studied in § 7 except that the 'bump' on S_1 faces the other way, so that the attitude of its reciprocal polar is reversed. The waves corresponding to S_2 lie within a wider cone N_2, of semi-angle 14·5°, and have no cusps, since the Gaussian curvature vanishes nowhere on S_2.

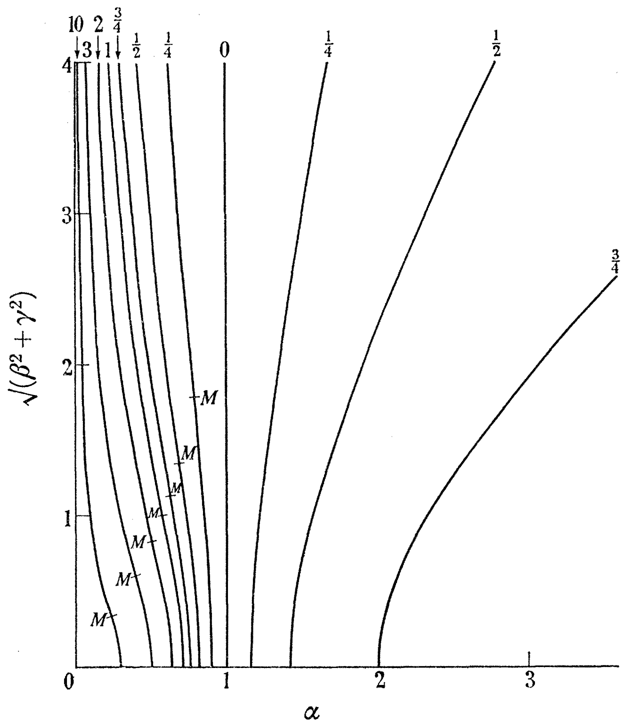

FIGURE 6. Hall-current effect on the shape of S. For ω/ω_i = 10, 3, 2, 1, $\frac{3}{4}$, $\frac{1}{2}$, $\frac{1}{4}$ and 0, S is obtained by rotating about the α-axis these curves together with their reflexions in the plane $\alpha = 0$. The unit of wave number is ω/a_1. The points of inflexion, marked M, become monoclastic curves on S after rotation.

Figure 8 shows how the semi-angles, θ_1 and θ_2, of the cones within which lie the wave systems corresponding to S_1 and S_2, vary with ω/ω_i. These graphs are obtained from the relationships

$$\frac{\omega}{\omega_i} = \frac{2\tan\theta_1}{(1-2\tan^{\frac{2}{3}}\theta_1)\,(1+\tan^{\frac{2}{3}}\theta_1)^{\frac{1}{2}}} = \sin\theta_2. \tag{116}$$

The above geometrical discussion gives a sufficient indication of how the Hall-current effect spreads out the waves in the case of a source of given frequency. Taking compressibility into account as well involves many more complications, including several more cuspidal edges, but makes little essential difference to the conclusions.

Since, however, the Hall-current effect renders the waves dispersive (the G of equation (112) being an inhomogeneous function of α, β, γ and ω, and the solution obtained being indeed quite different for different values of ω), we cannot at this stage say what will happen if a source of finite duration is applied instead of one varying sinusoidally with time. We would like to know how, if a source of finite duration is Fourier-analyzed with respect to time, the components of different frequencies spread out from the source.

The answer to this question is given by the theory of appendix B. This shows that the surfaces of constant phase for waves of a given frequency will be as obtained above, but that

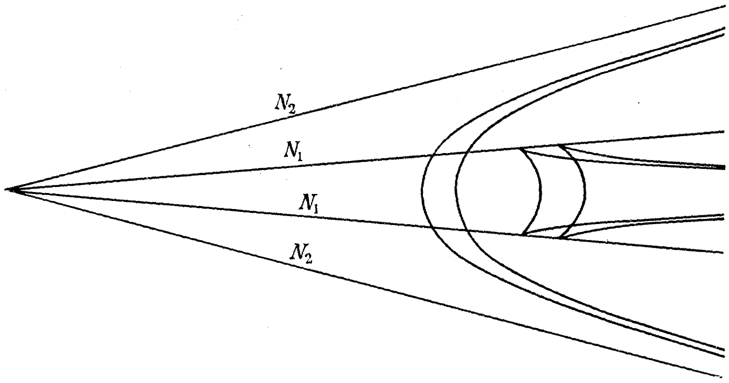

FIGURE 7. Hall-current effect on surfaces of constant phase. For $\omega/\omega_i = \frac{1}{4}$ this shows two surfaces of constant phase generated by S_1 and two generated by S_2 (only the part with $x > 0$ in each case). The phases, from left to right, are in the ratios 1:1·08:1:1·08.

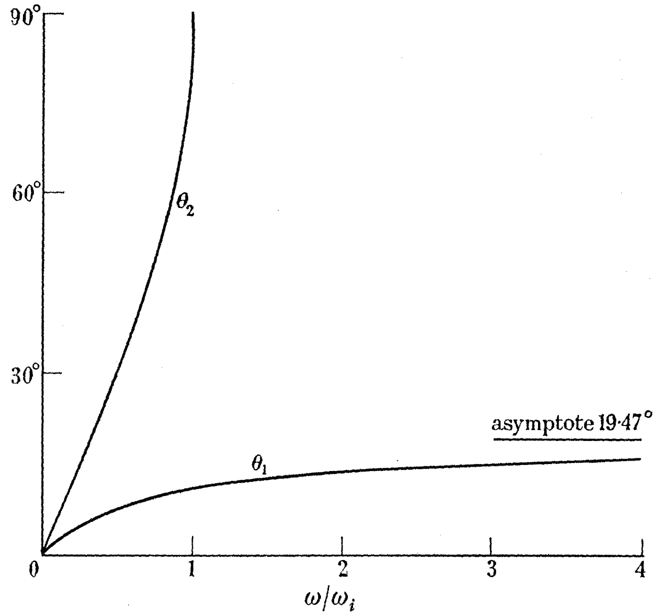

FIGURE 8. Variation with ω/ω_i of the semi-angles θ_1 and θ_2 of the cones N_1 and N_2.

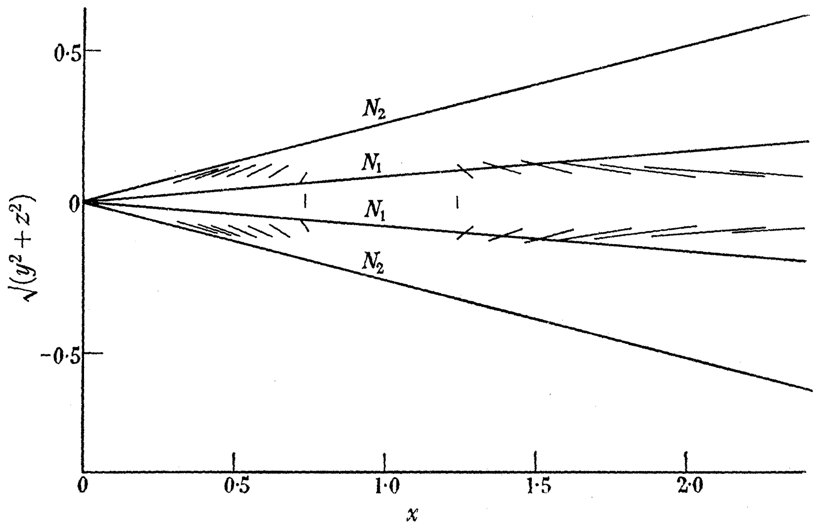

FIGURE 9. Dispersion due to Hall-current effect. For $\omega/\omega_i = \frac{1}{4}$ this shows the wave pattern at time t after the operation of a source of finite duration. The unit of distance is $a_1 t$.

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the distance of travel of any wave packet from the source after time t will be equal to t times its group velocity $\nabla G/(\partial G/\partial \omega)$. Applied to the problem of this section, with $\omega/\omega_i = \frac{1}{4}$, this gives a pattern of waves as in figure 9, with $a_1 t$ used as the unit of distance. Figure 9 is obtained from the fact that each wave element in figure 7 has travelled a distance from the origin equal to t times its velocity of energy propagation, while retaining its setting and spacing as given by the value of **k**. Figure 9 emphasizes the high degree of dispersion which the Hall-current effect may produce even at quite low frequencies.

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Appendix A. Velocity of energy propagation for a plane wave in an anisotropic medium

We suppose that, in a homogeneous conservative system, a plane wave

$$u = a \exp\left[\mathrm{i}(\omega t + \alpha x + \beta y + \gamma z)\right] \tag{A 1}$$

is a possible solution of the equations of motion for each (α, β, γ) in some region of wave-number space, provided that the frequency ω is a certain function of (α, β, γ) given by an equation

$$G(\alpha, \beta, \gamma, \omega) = 0. \tag{A 2}$$

The derivatives of ω with respect to α, β, γ are given by the equations

$$\frac{\partial \omega}{\partial \alpha} = -\frac{\partial G/\partial \alpha}{\partial G/\partial \omega}, \quad \frac{\partial \omega}{\partial \beta} = -\frac{\partial G/\partial \beta}{\partial G/\partial \omega}, \quad \frac{\partial \omega}{\partial \gamma} = -\frac{\partial G/\partial \gamma}{\partial G/\partial \omega}. \tag{A 3}$$

We now determine the x-component U_x of the velocity of energy propagation. This is

$$U_x = \frac{\text{energy crossing unit area of } x = 0}{\text{energy per unit volume}} = \frac{I}{E}, \quad \text{say.} \tag{A 4}$$

The energy E per unit volume, for given (α, β, γ), is a fixed multiple of the square of the wave amplitude, say

$$E = E_0 a^2. \tag{A 5}$$

To find I, we suppose that such additional (non-conservative) forces are applied as will attenuate the wave according to the law

$$u = a \exp\left[\mathrm{i}(\omega t + \alpha x + \beta y + \gamma z) - \tfrac{1}{2}\epsilon x\right], \tag{A 6}$$

where ϵ is very small compared with α, β and γ. Then the energy per unit volume at (x, y, z) becomes

$$E = E_0 a^2 \mathrm{e}^{-\epsilon x}, \tag{A 7}$$

and the total energy per unit area of the region $x > 0$ is

$$H = \int_0^\infty E\,\mathrm{d}x = \frac{E_0 a^2}{\epsilon}. \tag{A 8}$$

To find the required additional forces, note that the substitution of $\alpha + \frac{1}{2}\mathrm{i}\epsilon$ for α in (A 1), which yields (A 6), would produce a motion consonant with the equations of motion only if ω were simultaneously changed to

$$\omega + \tfrac{1}{2}\mathrm{i}\epsilon\,\partial\omega/\partial\alpha, \tag{A 9}$$

where $\partial\omega/\partial\alpha$ is given by (A 3). This would mean that, in the equation of motion of every particle of which the system is composed, the inertial force (minus the mass m times the acceleration $\ddot{\mathbf{r}}$) would change from $m\omega^2\mathbf{r}$ to

$$m\left(\omega + \tfrac{1}{2}\mathrm{i}\epsilon\frac{\partial\omega}{\partial\alpha}\right)^2 \mathbf{r} \doteqdot m\omega^2\mathbf{r} + \left(\epsilon\frac{\partial\omega}{\partial\alpha}\right) m\mathrm{i}\omega\mathbf{r}. \tag{A 10}$$

This would be exactly as if an additional force equal to $(\epsilon\,\partial\omega/\partial\alpha)$ times its momentum were applied to every particle.

Now, since the wave in $x > 0$ is attenuated, and has fixed total energy (A 8) per unit area, the sum of the rate of working of these additional forces, per unit area of the region $x > 0$, and I, the rate of energy transmission across unit area of $x = 0$, must be zero. Hence I is minus this rate of working, that is,

$$I = -\Sigma\left\{\left(\epsilon\frac{\partial\omega}{\partial\alpha}\right) m\dot{\mathbf{r}}\right\}.\dot{\mathbf{r}} = -\left(\epsilon\frac{\partial\omega}{\partial\alpha}\right) 2T, \tag{A 11}$$

where $T = \frac{1}{2}\Sigma m\dot{\mathbf{r}}^2$ is the total kinetic energy per unit area of the region $x > 0$. But in any progressive wave the averaged kinetic and potential energies are equal. Therefore $2T = H$, whence, by (A 11) and (A 8),

$$I = -\left(\epsilon\frac{\partial\omega}{\partial\alpha}\right) H = -E_0 a^2 \frac{\partial\omega}{\partial\alpha}, \tag{A 12}$$

and by (A 4), (A 5) and (A 3)

$$U_x = \frac{I}{E} = -\frac{\partial\omega}{\partial\alpha} = \frac{\partial G/\partial\alpha}{\partial G/\partial\omega}. \tag{A 13}$$

Similar results for U_y and U_z give finally

$$\mathbf{U} = \frac{\nabla G}{\partial G/\partial\omega}, \tag{A 14}$$

as stated in § 6.

In cases when ω is known as an explicit function of (α, β, γ) we can write more simply

$$\mathbf{U} = -\left(\frac{\partial\omega}{\partial\alpha}, \frac{\partial\omega}{\partial\beta}, \frac{\partial\omega}{\partial\gamma}\right), \tag{A 15}$$

a form which makes obvious the analogy with the one-dimensional case. The minus sign is present because only plus signs occur in (A 1).

Appendix B. Application of the methods of this paper to problems with zero initial conditions

In this appendix we depart from the exclusive discussion in the paper of harmonic time variation, to consider problems where the dependent variable u is everywhere zero until an initial instant, $t = 0$, after which a source begins to operate. This may be a source of finite, or even infinitesimal, duration; or it may be a sinusoidally varying source maintained for all $t > 0$. The latter case must yield, as $t \to \infty$, the solution satisfying the radiation condition, which thus will be determined by an approach alternative to that of § 6.

The equation to be solved is more general than (23), namely

$$P\left(\frac{\partial^2}{\partial t^2}, \frac{\partial^2}{\partial x^2}, \frac{\partial^2}{\partial y^2}, \frac{\partial^2}{\partial z^2}\right) u = \mathfrak{f}(x, y, z, t). \tag{B1}$$

Accordingly, a fourfold Fourier integral expression

$$\mathfrak{f} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left[\mathrm{i}(\alpha x+\beta y+\gamma z+\omega t)\right] \mathfrak{F}(\alpha,\beta,\gamma,\omega)\,\mathrm{d}\alpha\,\mathrm{d}\beta\,\mathrm{d}\gamma\,\mathrm{d}\omega \tag{B2}$$

for $\mathfrak{f}$ is necessary. Then, with

$$u = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left[\mathrm{i}(\alpha x+\beta y+\gamma z+\omega t)\right] \mathfrak{U}(\alpha,\beta,\gamma,\omega)\,\mathrm{d}\alpha\,\mathrm{d}\beta\,\mathrm{d}\gamma\,\mathrm{d}\omega, \tag{B3}$$

we have

$$G\mathfrak{U} = \mathfrak{F} \tag{B4}$$

(just as at the beginning of § 3), and we have to pick the solution such that $u = 0$ for $t < 0$. (German type is used in this appendix for four-dimensional analogues of three-dimensional quantities appearing in the paper.)

Now we know that the source strength $\mathfrak{f} = 0$ for $t < 0$, whence its transform,

$$\mathfrak{F} = \frac{1}{16\pi^4}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left[-\mathrm{i}(\alpha x+\beta y+\gamma z+\omega t)\right] \mathfrak{f}(x,y,z,t)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}t, \tag{B5}$$

is a regular function of ω (for each α, β, γ) throughout the lower half of the complex ω-plane; for in (B5) the integral with respect to t is the same as one from 0 to ∞ and therefore converges uniformly when $\mathfrak{J}\omega \leqslant -\epsilon$ (so that $|\mathrm{e}^{-\mathrm{i}\omega t}| \leqslant \mathrm{e}^{-\epsilon t}$) for any $\epsilon > 0$. On the other hand, $\mathfrak{F}$ may have singularities for real ω, which will correspond as described by Lighthill (1958) to the behaviour of $\mathfrak{f}$ as $t \to +\infty$. One can avoid these singularities in the integral (B2) by taking the path of integration with respect to ω slightly below the real axis, for example, along a line $\mathfrak{J}\omega = -\epsilon$. Note that the $\mathfrak{f}$ so produced *does* vanish for $t < 0$, since the regularity of $\mathfrak{F}$ in the lower half of the ω-plane means that the path of integration can be deformed into a large semi-circle in that region, the integral over which vanishes for $t < 0$ by Jordan's lemma.

We now apply similar considerations to u and $\mathfrak{U}$, but until the last paragraph we treat only the case when there is no complex value of ω for which G vanishes (with real α, β, γ). Then we can obtain a solution for u which is zero for $t < 0$ by putting $\mathfrak{U} = \mathfrak{F}/G$ (from (B4)), and integrating on $\mathfrak{J}\omega = -\epsilon$ in (B3). This is because $\mathfrak{U}$, like $\mathfrak{F}$, has then no singularities in the lower half of the ω-plane, and so the Jordan-lemma argument can still be applied. The resulting solution,

$$u = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\mathrm{i}\epsilon-\infty}^{-\mathrm{i}\epsilon+\infty} \exp\left[\mathrm{i}(\alpha x+\beta y+\gamma z+\omega t)\right] \frac{\mathfrak{F}}{G}\,\mathrm{d}\alpha\,\mathrm{d}\beta\,\mathrm{d}\gamma\,\mathrm{d}\omega, \tag{B6}$$

would also be obtained if a Laplace-transform approach were used.

The asymptotic form of (B6) is next obtained as $r \to \infty$ and $t \to \infty$ simultaneously; more precisely, we let the point (x, y, z, t) tend to infinity along a line $\mathfrak{l}$ in four-dimensional space-time whose inclination to the time-axis is less than $\frac{1}{2}\pi$. The method of § 3 is followed very closely, the main difference being merely the extra dimension.

Rotating the axes so that the x-axis is along $\mathfrak{l}$ gives

$$u = \iiint \mathrm{d}\beta\,\mathrm{d}\gamma\,\mathrm{d}\omega \int \frac{\mathfrak{F}}{G} \mathrm{e}^{\mathrm{i}\alpha x}\,\mathrm{d}\alpha, \tag{B 7}$$

the new α being a linear combination of the old α, β, γ and ω which has positive coefficient of ω. It follows that the path of integration with respect to the new α lies slightly below the real axis.

We now suppose first that $\mathfrak{f}(x, y, z, t)$ vanishes outside a bounded region of space-time, so that the source is of finite duration. Then $\mathfrak{F}$ possesses no singularities on the real axis, and the only singularities of the integrand are zeros of G. In this case we estimate the inner integral in (B 7) by shifting the path of integration to $\mathfrak{J}\alpha = +h$ (which renders it $O(\mathrm{e}^{-\alpha x})$ and hence negligible), and are left with $2\pi\mathrm{i}$ times the sum of the residues of the integral at the zeros of G. Thus,

$$u = 2\pi\mathrm{i} \iiint_{\mathfrak{S}} \frac{\mathfrak{F}}{G_\alpha} \mathrm{e}^{\mathrm{i}\alpha x}\,\mathrm{d}\beta\,\mathrm{d}\gamma\,\mathrm{d}\omega + O\,(x^{-N}), \tag{B 8}$$

where the integral is over the whole hypersurface $\mathfrak{S}$ whose equation is $G = 0$.

The integral (B 8) is next estimated by the method of stationary phase as a sum of contributions from points on $\mathfrak{S}$ where the normal is in the x-direction. If these are all points where the principal curvatures (say, after rotation of axes, κ_β, κ_γ and κ_ω) are non-zero, we obtain, as the contribution from the mth point $(\alpha_m, \beta_m, \gamma_m, \omega_m)$,

$$u_m = 2\pi\mathrm{i} \frac{\mathfrak{F}}{G_\alpha} \mathrm{e}^{\mathrm{i}\alpha_m x} \prod_{\beta,\gamma,\omega} \left\{ \left(\frac{2\pi}{x\,|\kappa_\beta|} \right)^{\frac{1}{2}} \exp\left[\tfrac{1}{4}\pi\mathrm{i}\,\mathrm{sgn}\,\kappa_\beta\right] \right\}. \tag{B 9}$$

In a form invariant under rotation of axes, (B 9) becomes

$$u_m = \frac{(2\pi)^{\frac{5}{2}}}{\mathfrak{r}^{\frac{3}{2}}} \left\{ \frac{C\mathfrak{F} \exp\left[\mathrm{i}(\alpha x + \beta y + \gamma z + \omega t)\right]}{|\Box G|\,\sqrt{|\mathfrak{K}|}} \right\}_{\substack{\alpha=\alpha_m,\,\beta=\beta_m,\\ \gamma=\gamma_m,\,\omega=\omega_m}}. \tag{B 10}$$

Here $\mathfrak{r} = \sqrt{(x^2+y^2+z^2+t^2)}$, which is $r(1+v^{-2})^{\frac{1}{2}}$ if v is the ratio of r to t on $\mathfrak{l}$; C is a phase factor of modulus 1; $\Box G$ is the gradient of G in Cartesian 4-space; and $\mathfrak{K}$ is the Gaussian curvature of the hypersurface $\mathfrak{S}$, which can be expressed in an invariant form similar to that noted at the end of § 3—each element of the matrix of products of first derivatives of G being multiplied into the corresponding co-factor of the matrix of second derivatives, and divided by $|\Box G|^5$. The solution u is asymptotically a sum of terms (B 10) over all points on $\mathfrak{S}$ where the normal to the hypersurface is parallel to $\mathfrak{l}$, provided $\mathfrak{K}$ is non-zero at each. No other principal of selection (like that of § 6) governs the choice of points.

The condition that $\mathfrak{l}$ be normal to $\mathfrak{S}$ can be written

$$\mathbf{r}{:}\nabla G = t{:}\,\frac{\partial G}{\partial \omega}, \tag{B 11}$$

which shows that $\mathbf{r}/t$ is the energy propagation velocity $\nabla G/(\partial G/\partial\omega)$. This emphasizes again that energy spreads out radially from the source with this velocity.

The attenuation like $r^{-\frac{3}{2}}$ along any radius vector is characteristic of pulses propagated three-dimensionally outward in a dispersive medium. For the pulse necessarily contains elements in a range of frequencies, and therefore with a range of wave speeds even for a fixed direction. The region of disturbance therefore grows like a hollow sphere (possibly flattened in shape) whose internal and external radii increase in proportion, so that its volume increases as r^3, the energy density decreases as r^{-3} and the amplitude as $r^{-\frac{3}{2}}$.

If, however, the waves are non-dispersive, G is homogeneous, so that the hypersurface $\mathfrak{S}$ given by $G = 0$ is a hypercone, that is, a *developable* hypersurface, with $\mathfrak{K}$ identically zero. Its generators are lines $\mathfrak{L}$ joining the origin of space-time to points of the surface S, which is given by $G = 0$ for fixed ω. A hyperplane $\mathfrak{P}$ touching $\mathfrak{S}$ does so along a whole such line $\mathfrak{L}$. One finds then an asymptotic contribution (compare § 5) of order $\mathfrak{r}^{-1}$ along all such radii as are normal to one of these hyperplanes $\mathfrak{P}$, but only along these; furthermore, the contribution comes from the whole of the line of tangency $\mathfrak{L}$.

The physical explanation is that for non-dispersive waves the disturbance due to a pulse at time $t = 0$ is propagated at a fixed speed in each spatial direction, components of all frequencies travelling at the same speed (or possibly there might be a finite set of speeds, each being the energy propagation velocity for waves whose fronts are in a particular direction). Hence the *radial* extent of the disturbance does not increase; its volume increases as r^2, its energy density decreases as r^{-2} and its amplitude as r^{-1}.

After this discussion of sources of finite duration in the dispersive and non-dispersive cases, we note the results for a harmonically oscillating source of frequency Ω started at $t = 0$. Then

$$\mathfrak{f}(x, y, z, t) = f(x, y, z)\, e^{i\Omega t} H(t), \tag{B 12}$$

with $H(t) = 1$ for $t > 0$ and 0 for $t < 0$. This gives

$$\mathfrak{F}(\alpha, \beta, \gamma, \omega) = \frac{F(\alpha, \beta, \gamma)}{2\pi i(\omega - \Omega)} \tag{B 13}$$

throughout the lower half of the complex ω-plane. With this $\mathfrak{F}$ the asymptotic evaluation of (B 6) can be carried out as follows. Take first a fixed value of ω (with imaginary part $-\epsilon$); then asymptotically integrate with respect to α, β and γ as in § 6 (where also a frequency with imaginary part $-\epsilon$ was used), leading to the result of theorem 2. Next, divide by $2\pi i$ $(\omega - \Omega)$ and integrate with respect to ω. As $t \to +\infty$ this replaces ω by Ω, the given frequency.

Thus we retrieve the conclusions of theorem 2, regarding the solution satisfying the radiation condition, if we approach this solution by 'switching on and waiting'. This might have been expected, since the $e^{\epsilon t}$ factor inserted in the source strength to obtain theorem 2 could be regarded as a particular, somewhat languid, method of 'switching on and waiting'.

We conclude this appendix by taking up the possibility (excluded at the beginning) that $G = 0$ for some complex ω. For given real α, β, γ such ω would occur in conjugate pairs, with one ω from each pair situated in the lower half of the complex plane. Then to get $u = 0$ for $t < 0$ it is necessary to take the path of integration with respect to ω in (B 6) below all these poles of the integrand as well as below those on the real axis. Therefore, when the path is deformed into the real axis and beyond to carry out the procedures described above, there are in general residues from these poles to be taken into account; these increase exponentially with time. Thus, whenever the source includes Fourier components with wave numbers in the range for which there exist solutions of $G = 0$ (that is, plane waves) of complex frequency, an exponentially increasing disturbance will ensue. This is the mathematical condition corresponding to a physical system unstable to disturbances in this range of wave numbers. The systems discussed in earlier paragraphs of this appendix were stable systems without complex-frequency solutions of $G = 0$.